# MINIMAL REPRESENTATIONS OF SIMPLE REAL LIE GROUPS OF HERMITIAN TYPE - THE FOCK MODEL

Dehbia ACHAB

**Abstract** In the recent paper [Achab 2012], a unified geometric realization to the minimal representations of simple real Lie groups of non Hermitian type is given, based on the geometric setting introduced in [Achab 2011] and the analysis of the Brylinski-Kostant model, given in [Achab-Faraut 2012]. We give in this paper a unified geometric realization to the minimal representations of simple real Lie groups of Hermitian type.

*Key words*: Minimal representation, Lie algebra, Jordan algebra, Conformal group,

*Mathematics Subject Index 2010*:17C36, 22E46, 32M15, 33C80.

**Introduction**. —

In the paper [Achab, 2011] the following principles were established:

1) Given a complex simple Jordan algebra $V$ of rank $r$ and dimension $n$, and a homogeneous polynomial $Q$ of degree $2r$ on $V$ ($Q(v) = \Delta(v)^2$ where $\Delta$ is the determinant on $V$) there is a covering $\tilde{K}$of order one or two, of the conformal group $Conf(V, Q)$ and a cocycle $\mu : \tilde{K} \times V \to \mathbb{C}$ such that the corresponding cocycle representation of $\tilde{K}$ on the polynomial functions on $V$ leaves the space $\mathfrak{p}$, spanned by $Q$ and its translates $z \mapsto Q(z - a)$ with $a \in V$, invariant, producing an irreducible representation $\kappa$ of $\tilde{K}$ on $\mathfrak{p}$ (see Proposition 1.1 and Corollary 1.2 in [Achab 2011]). In particular, the group $L$, which is the preimage of the structure group of $V$ by the covering, acts on $\mathfrak{p}$ by the restriction of $\kappa$.

2) There is $\tilde{H} \in \mathfrak{z}(\mathfrak{l})$ the center of $\mathfrak{l} = Lie(L)$ such that $d\kappa(\tilde{H})$ defines a grading of $\mathfrak{p}$ given by

$$\mathfrak{p} = \mathfrak{p}_{-r} \oplus \mathfrak{p}_{-r+1} \oplus \ldots \oplus \mathfrak{p}_0 \oplus \ldots \oplus \mathfrak{p}_{r-1} \oplus \mathfrak{p}_r$$

where $\mathfrak{p}_i = \{p \in \mathfrak{p} \mid d\kappa(\tilde{H})p = ip\}$ is the set of homogeneous polynomials of degree $i + r$ in $\mathfrak{p}$. In particular, $\mathfrak{p}_{-r} = \mathbb{C}, \mathfrak{p}_r = \mathbb{C} \cdot Q, \mathfrak{p}_{-r+1} \simeq \mathfrak{p}_{r-1} \simeq V$ are simple $\mathfrak{l}$-modules and more precisely, when $r \neq 1$, $\mathcal{V} := \mathfrak{p}_{-r+1}$ is the dual of $V$ and $\mathcal{V}^\sigma := \mathfrak{p}_{r-1} = \kappa(\sigma)\mathfrak{p}_{-r+1}$, where $j$ is the conformal inversion on $V$ and $\sigma$ its preimage in $\tilde{K}$ by the covering map $s : \tilde{K} \to \mathrm{Conf}(V, Q)$. In the special case $r = 1$,we denote by $\mathcal{V} = \mathfrak{p}_{-r}$ and $\mathcal{V}^\sigma := \mathfrak{p}_r$.

3) $Lie(L) \oplus \mathcal{V} \oplus \mathcal{V}^\sigma$ carries the structure of a complex simple Lie algebra $\mathfrak{g}$ (see Theorem 8.1 in [Achab 2011]). One constructs a non compact real form $\mathfrak{g}_\mathbb{R}$ of $\mathfrak{g}$, of Hermitian type, starting with a Euclidean real form $V_\mathbb{R}$ of $V$. The construction also yields a compact real form $L_\mathbb{R}$ of the group $L$. It turns out that we obtain in this way (see table 3 in [Achab 2011]) all the simple real Lie algebras of Hermitian type which possess a strongly minimal real nilpotent orbit, i.e. such that $\mathcal{O}_{\min} \cap \mathfrak{g}_\mathbb{R} \neq \emptyset$, where $\mathcal{O}_{\min}$ is the minimal nilpotent adjoint orbit of $\mathfrak{g}$. Recall that from a point of view of representation theory, the condition that $\mathcal{O}_{\min} \cap \mathfrak{g}_\mathbb{R} \neq \emptyset$ is natural, since it is a necessary condition for a simple real Lie group with Lie algebra $\mathfrak{g}_\mathbb{R}$ to admit an irreducible unitary representation with associated complex nilpotent orbit $\mathcal{O}_{min}$ (see Theorem 8.4 in [Vogan, 1991]).

Recall that when $V$ is simple, the map $v \mapsto \mathrm{trace}(v)$ is (up to a scalar) the only linear form on $V$ which is invariant under the group $Aut(V)$ of automorphisms of $V$. Let $\tau \in \mathfrak{p}_{-r+1}$ be this linear form and denote by $\tau^\sigma = \kappa(\sigma)\tau \in \mathfrak{p}_{r-1}$, and observe that for $v \in V$, $\tau(v) = tr(v)$ and $\tau^\sigma(v) = \Delta(v)^2 tr(-v^{-1})$. In the Lie algebra structure of $\mathfrak{g} = Lie(L) \oplus \mathcal{V} \oplus \mathcal{V}^\sigma$, one has

$$[\tilde{H}, \tau] = (1 - r)\tau \quad , \quad [\tilde{H}, \tau^\sigma] = -(1 - r)\tau^\sigma \quad [\tilde{\tau}, \tilde{\tau}^\sigma] = \tfrac{1}{2}\tilde{H}.$$

The two $L$-orbits $\Xi := L.\tau \subset \mathcal{V}$ and $\Xi^\sigma := L.\tau^\sigma \subset \mathcal{V}^\sigma$ are conical varieties , related by $\Xi^\sigma = \kappa(\sigma)\Xi$ and correspond respectively to $\mathcal{O}_{\min} \cap \mathcal{V}$ and $\mathcal{O}_{\min} \cap \mathcal{V}^\sigma$ in such a way that $\mathcal{O}_{\min} \cap \mathcal{W} = \Xi \cup \Xi^\sigma$. They admit a coordinate system

$$X := \{P(a) \mid a = \sum_{i=1}^{r} a_i c_i \mid a_i \in \mathbb{C}, \mathrm{Re}(a_1) \geq \ldots \geq \mathrm{Re}(a_r) > 0, \mathrm{Im}(a_1) \geq \ldots \geq \mathrm{Im}(a_r) > 0\}$$

where $\{c_1, \ldots, c_r\}$ is a complete system of orthogonal primitive idempotents in $V_{\mathbb{R}}$.

The map $\tilde{\pi}(\sigma) : \mathcal{O}(\Xi^\sigma) \to \mathcal{O}(\Xi), f^\sigma \mapsto \tilde{\pi}(\sigma) f^\sigma$, where $\tilde{\pi}(\sigma) f^\sigma(\xi) = f^\sigma(\kappa(\sigma)\xi)$ is an isomorphism. In the coordinate system, it is given by:

$$\tilde{\pi}(\sigma) : \mathcal{O}(X) \to \mathcal{O}(X), \phi(P(a)) \mapsto \phi(P(a^{-1})).$$

The $L$-action on $\Xi$ and on $\Xi^\sigma$ yield representations $\pi_\alpha$ and $\pi_\alpha^\sigma$ of $L$ on the spaces $\mathcal{O}(\Xi)$ and $\mathcal{O}(\Xi^\sigma)$ of holomorphic functions and then representations $\tilde{\pi}_\alpha$ and $\tilde{\pi}_\alpha^\sigma$ of $L$ on the space $\mathcal{O}(X)$. The isomorphism $\tilde{\pi}(\sigma)$ intertwines these representations.

Along the paper, The rank $r$ will be assumed $> 1$. The case $r = 1$ will be considered separately, at the last section.

The subspaces $\mathcal{O}_{-\frac{m}{(r-1)}}(\Xi)$ (resp. $\mathcal{O}_{-\frac{m}{(r-1)}}(\Xi^\sigma)$) of fixed degree of homogeneity $-\frac{m}{(r-1)}$ in the fiber direction are irreducible $L$-modules. Homogeneity in the fiber direction allows to identify the space $\mathcal{O}_{-\frac{m}{(r-1)}}(\Xi)$ (resp. $\mathcal{O}_{-\frac{m}{(r-1)}}(\Xi^\sigma)$) with the subspace $\mathcal{O}_m(X) \subset \mathcal{O}_{2m}(\mathbb{C}^r)$ of holomorphic functions on $X$, homogeneous of degree $m$. Also, the subspaces $\mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi)$ (resp. $\mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma)$) of fixed degree of homogeneity $-\frac{m}{(r-1)} - \frac{1}{2(r-1)}$ in the fiber direction are irreducible $L$-modules. Homogeneity in the fiber direction allows to identify $\mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi)$ (resp. $\mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma)$) with the subspace $\mathcal{O}_{m+\frac{1}{2}}(X) \subset \mathcal{O}_{2m+1}(\mathbb{C}^r)$ of holomorphic functions on $X$, homogeneous of degree $m + \frac{1}{2}$.

We consider the subspace $\widetilde{\mathcal{O}}(X)$ of $\mathcal{O}(X)$ of the functions $\phi$ such that $\phi \circ P$ extends to a holomorphic function on $\mathbb{C}^r$. And then consider the corresponding subspaces $\widetilde{\mathcal{O}}(\Xi)$ and $\widetilde{\mathcal{O}}(\Xi^\sigma)$ of $\mathcal{O}(\Xi)$ and $\mathcal{O}(\Xi^\sigma)$ and denote by

$$\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}}(\Xi) := \mathcal{O}_{-\frac{m}{(r-1)}}(\Xi) \cap \widetilde{\mathcal{O}}(\Xi),$$
$$\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi) := \mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi) \cap \widetilde{\mathcal{O}}(\Xi),$$

and similarly,

$$\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}}(\Xi^\sigma) := \mathcal{O}_{-\frac{m}{(r-1)}}(\Xi^\sigma) \cap \widetilde{\mathcal{O}}(\Xi^\sigma),$$
$$\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma) := \mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma) \cap \widetilde{\mathcal{O}}(\Xi^\sigma).$$

It turns out that for $m < 0$

$$\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}}(\Xi) = \widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi) = \{0\}$$

and

$$\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}}(\Xi^\sigma) = \widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma) = \{0\}.$$

There exist $L_{\mathbb{R}}$-invariant inner products on the spaces

$$\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}}(\Xi) \simeq \widetilde{\mathcal{O}}_m(X) \text{ and } \widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}}(\Xi^\sigma) \simeq \widetilde{\mathcal{O}}_m(X) \ (\ m \geq 0),$$

which have reproducing kernels $H^{2m+r+1}$ and $H_\sigma^{\alpha(m)}$, and, there exist $L_{\mathbb{R}}$-invariant inner products on the spaces

$$\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi) \simeq \widetilde{\mathcal{O}}_{m+\frac{1}{2}}(X) \text{ and } \widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma) \simeq \widetilde{\mathcal{O}}_{m+\frac{1}{2}}(X) \ (m \geq 0),$$

which have reproducing kernels $H^{2m+2+r}$ and $H_\sigma^{\alpha(m+\frac{1}{2})}$, where

$$H(z) = \tau(\tfrac{e}{r} + z\bar{z}) \text{ and } H_\sigma(z) = \tau^\sigma(\tfrac{e}{r} + z\bar{z}).$$

Adding the kernels $H^{2m+r+1}$ and $H^{2m+2+r}$ (resp. $H_\sigma^{\alpha(m)}$ and $H_\sigma^{\alpha(m+\frac{1}{2})}$) with arbitrarily chosen non-negative weights yields a multiplicity free unitary $L_{\mathbb{R}}$-representation on a Hilbert subspace of $\mathcal{O}(\Xi)$ ( respectively $\mathcal{O}(\Xi^\sigma)$).

For the representation $\rho$, the polynomials $p \in \mathcal{W}$ act by multiplication and the polynomials $p \in \mathcal{W}^\sigma$ act by differentiation on the spaces of finite sums

$$\mathcal{O}_{\text{fin}}(\Xi) := \mathcal{O}_{\text{odd}}(\Xi) \oplus \mathcal{O}_{\text{even}}(\Xi)$$

where

$$\mathcal{O}_{\text{even}}(\Xi) := \sum_{m\in\mathbb{N}} \widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}}(\Xi) = \sum_{m\in\mathbb{N}} \widetilde{\mathcal{O}}_m(X)$$

and

$$\mathcal{O}_{\text{odd}}(\Xi) := \sum_{m\in\mathbb{N}} \widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi) = \sum_{m\in\mathbb{N}} \widetilde{\mathcal{O}}_{m+\frac{1}{2}}(X).$$

Similarly, for the representation $\rho^\sigma := \tilde{\pi}(\sigma)\rho\tilde{\pi}(\sigma)$, the polynomials $p \in W^\sigma$ act by multiplication and the polynomials $p \in \mathcal{W}^\sigma$ act by differentiation on the spaces of finite sums

$$\mathcal{O}_{\text{fin}}(\Xi^\sigma) := \mathcal{O}_{\text{odd}}(\Xi^\sigma) \oplus \mathcal{O}_{\text{even}}(\Xi^\sigma)$$

where

$$\mathcal{O}_{\text{even}}(\Xi^\sigma) := \sum_{m\in\mathbb{N}} \widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}}(\Xi^\sigma) = \sum_{m\in\mathbb{N}} \widetilde{\mathcal{O}}_m(X)$$

and

$$\mathcal{O}_{\text{odd}}(\Xi^\sigma) := \sum_{m\in\mathbb{N}} \widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma) = \sum_{m\in\mathbb{N}} \widetilde{\mathcal{O}}_{m+\frac{1}{2}}(X).$$

In particular, for $E := \tau$ and $F := \tau^\sigma$ in $\mathcal{W}$, the operators $\rho(E), \rho(F), \rho^\sigma(E), \rho^\sigma(F)$ are $L$-equivariant.

Since

$$[\tilde{H}, E] = (1-r)E \quad , \quad [\tilde{H}, F] = (r-1)F \quad , \quad [E, F] = \tfrac{1}{2}\tilde{H}$$

then, for $H := \frac{2}{r-1}\tilde{H}$, we have

$$[H, E] = -2E \quad , \quad [H, F] = 2F \quad , \quad [E, F] = \tfrac{r-1}{4}H$$

and the number $\alpha$ is chosen such that $[\rho(E), \rho(F)] = \frac{r-1}{4} d\pi_\alpha(H)$. This allows to the maps

$$\rho : \mathcal{W} \to End(\mathcal{O}_{\text{fin}}(\Xi)), p \mapsto \rho(p)$$

and

$$\rho^\sigma : \mathcal{W} \to End(\mathcal{O}_{\text{fin}}(\Xi^\sigma)), p \mapsto \rho^\sigma(p)$$

to complement $d\pi_\alpha$ and $d\pi_\alpha^\sigma$ to representations $d\pi_\alpha + \rho$ and $d\pi_\alpha^\sigma + \rho^\sigma$ of $\mathfrak{g} = Lie(L) + \mathcal{W}$ on $\mathcal{O}_{\text{fin}}(\Xi)$ and $\mathcal{O}_{\text{fin}}(\Xi^\sigma)$ respectively .

Furthermore, it is possible to determine a set of weights such that the restrictions of $d\pi_\alpha + \rho$ and $d\pi_\alpha^\sigma + \rho^\sigma$ to $\mathfrak{g}_\mathbb{R}$ are infinitesimally unitary. Moreover, Nelson's criterion can be used to show that these representations integrate to the simply connected Lie group $\widetilde{G_\mathbb{R}}$ with Lie algebra $\mathfrak{g}_\mathbb{R}$.

The two representations so obtained are realized on Hilbert spaces $\mathcal{H}$ and $\mathcal{H}^\sigma$ of holomorphic functions on $\Xi$ and $\Xi^\sigma$. The reproducing kernel of $\mathcal{H}$ is given by $H(z, z')^{r+1}\exp(H(z, z'))$. The space $\mathcal{H}$ is a weighted Bergman space. The norm is given by an integral on $\mathbb{C}^r$ with a weight given by $p(z) = H(z)^{r+1}\exp(-H(z))$.

The recent work by J. Hilgert,T. Kobayashi, J. Mollers and B. Orsted, (see [HKMO, 2012]), consists in constructing with a different method, Schrodinger and Fock models for minimal representations of Hermitian real Lie groups of tube type and the authors obtained a nice formula for the intertwinning operator, the Bargmann transform, between the two models.

Our method works for all simple Hermitian real Lie groups which admit minimal representations. It leads to some 'universal' Fock space. Since it is based on the same theory than that considered in the case of simple real Lie groups of non Hermitian type, we can consider that the papers [Achab 2012] and the present one give a unified theory for the 'universal' Fock model of the minimal representations of simple real Lie groups.

Our theory of 'universal' Fock models for minimal representations of simple real Lie groups allows to obtain first the classical Fock models by using the Cayley transform, and second the Schrodinger models by restriction principle of holomorphic functions to natural suitable real spaces. This view point will lead to a bridge between our theory and the [HKMO, 2012] theory. This will be the subject of a next paper.

**1. Construction process of complex simple Lie algebras and of simple real Lie algebras of Hermitian type**. —

Let $V$ be a simple complex Jordan algebra with rank $r$ and dimension $n$ and $Q$ the homogeneous polynomial of degree $2r$ on $V$ given by $Q(v) = \Delta(v)^2$ where $\Delta$ is the Jordan algebra determinant. Let

$$\mathrm{Str}(V, Q) = \{g \in GL(V) \mid \exists \gamma(g) \in \mathbb{C}, Q(g \cdot x) = \gamma(g)Q(x)\}.$$

The conformal group $\mathrm{Conf}(V, Q)$ is the group of rational transformations $g$ of $V$ generated by: the translations $z \mapsto z + a$ $(a \in V)$, the dilations $z \mapsto \ell \cdot z$ $(\ell \in L)$, and the conformal inversion $j : z \mapsto -z^{-1}$.

Let $\mathfrak{p}$ be the space of polynomials on $V$ generated by the translated $Q(z - a)$ of $Q$, with $a \in V$. Let $\kappa$ be the cocycle representation of $\mathrm{Conf}(V, Q)$ or of a covering $\tilde{K}$ of order two of it on $\mathfrak{p}$, defined in [A11] and [AF12] as follows:

*Case 1*

In case there exists a character $\chi$ of $\mathrm{Str}(V, Q)$ such that $\chi^2 = \gamma$, then let $\tilde{K} = \mathrm{Conf}(V, Q)$. Define the cocycle

$$\mu(g, z) = \chi((Dg(z)^{-1}) \quad (g \in \tilde{K},\ z \in V),$$

and the representation $\kappa$ of $\tilde{K}$ on $\mathfrak{p}$,

$$(\kappa(g)p)(z) = \mu(g^{-1}, z)p(g^{-1} \cdot z).$$

The function $\kappa(g)p$ belongs actually to $\mathfrak{p}$ (see [Faraut-Gindikin, 1996], Proposition 6.2). The cocycle $\mu(g, z)$ is a polynomial in $z$ of degree $\leq \deg Q$ and

$$\begin{aligned}
(\kappa(\tau_a)p)(z) &= p(z - a) \quad (a \in V),\\
(\kappa(\ell)p)(z) &= \chi(\ell)p(\ell^{-1} \cdot z) \quad (\ell \in L),\\
(\kappa(j)p)(z) &= Q(z)p(-z^{-1}).
\end{aligned}$$

*Case 2* Otherwise the group $\tilde{K}$ is defined as the set of pairs $(g, \mu)$ with $g \in \mathrm{Conf}(V, Q)$, and $\mu$ is a rational function on $V$ such that

$$\mu(z)^2 = \gamma(Dg(z))^{-1}.$$

We consider on $\tilde{K}$ the product $(g_1, \mu_1)(g_2, \mu_2) = (g_1 g_2, \mu_3)$ with $\mu_3(z) = \mu_1(g_2 \cdot z)\mu_2(z)$. For $\tilde{g} = (g, \mu) \in \tilde{K}$, define $\mu(\tilde{g}, z) := \mu(z)$. Then $\mu(\tilde{g}, z)$ is a cocycle: $\mu(\tilde{g}_1\tilde{g}_2, z) = \mu(\tilde{g}_1, \tilde{g}_2 \cdot z)\mu(\tilde{g}_2, z)$, where $\tilde{g} \cdot z = g \cdot z$ by definition.

Recall that the representation $\kappa$ of $\tilde{K}$ on $\mathfrak{p}$ is irreducible and

$$\big(\kappa(g)p\big)(z) = \mu(g^{-1}, z)p(g^{-1} \cdot z).$$

Observe that since the degree of $Q$ is even it follows that the inverse in $\tilde{K}$ of $\sigma = (j, Q(z))$ is $\sigma$.

Let $L$ be $\mathrm{Str}(V, Q)$ in the first case or the preimage of $\mathrm{Str}(V, Q)$ by the covering map $s : \tilde{K} \to \mathrm{Conf}(V, Q)$., in the second case. It is established in [A11] that there is $\tilde{H} \in \mathfrak{z}(\mathfrak{l})$, where $\mathfrak{l} = Lie(L)$ which defines a grading of $\mathfrak{p}$:

$$\mathfrak{p} = \mathfrak{p}_{-r} \oplus \mathfrak{p}_{-r+1} \oplus \ldots \oplus \mathfrak{p}_0 \oplus \ldots \oplus \mathfrak{p}_{r-1} \oplus \mathfrak{p}_r,$$

where

$$\mathfrak{p}_j = \{p \in \mathfrak{p} \mid d\kappa(\tilde{H})p = jp\}$$

is the set of polynomials in $\mathfrak{p}$, homogeneous of degree $j + r$. Furthermore $\kappa(\sigma) : \mathfrak{p}_j \to \mathfrak{p}_{-j}$, and

$$\mathfrak{p}_{-r} = \mathbb{C}, \quad \mathfrak{p}_r = \mathbb{C}\, Q, \quad \mathfrak{p}_{r-1} \simeq V, \quad \mathfrak{p}_{-r+1} \simeq V.$$

Observe that $\mathfrak{p}_{-r+1}$ is the space of linear forms on $V$, $\mathfrak{p}_{r-1} = \{\kappa(\sigma)\tau \mid \tau \in \mathfrak{p}_{-r+1}\}$ and for a linear form $\tau$ on $V$,

$$\kappa(\sigma)\tau(z) = \Delta(z)^2\tau(-z^{-1}).$$

Assume $r \neq 1$ and denote by $\mathcal{W} = \mathcal{V} \oplus \mathcal{V}^\sigma$ where

$$\mathcal{V} = \mathfrak{p}_{-r+1} \quad , \quad \mathcal{V}^\sigma = \mathfrak{p}_{r-1}.$$

Then (see Theorem 8.1 in [A11]) $\mathfrak{g} := \mathfrak{l} \oplus \mathcal{W}$ carries a unique simple Lie algebra structure such that ,

$$\begin{aligned} &(i) \quad [X, X'] = [X, X']_{\mathfrak{k}} \quad (X, X' \in \mathfrak{l}), \\ &(ii) \quad [X, p] = d\kappa(X)p \quad (X \in \mathfrak{l}, p \in \mathcal{W}). \end{aligned}$$

In fact, for every $p \in \mathcal{V}$, there is a unique $X_p \in \mathfrak{k}_{-1}$ such that $p = d\kappa(X_p)Q$ and for every $p^\sigma \in \mathcal{V}^\sigma$, there is a unique $X_{p^\sigma} \in \mathfrak{k}_1$ such that $p^\sigma = d\kappa(X_{p^\sigma})1$ (see Lemma 1.1 in [A11]) and one defines the bracket $[p, p^\sigma] := [X_p, X_{p^\sigma}]$.

Let's consider the linear form $\tau : z \mapsto tr(z) = \mathrm{trace}(z)$, which is the unique (up to a scalar) $K$-invariant linear form on $V$, with $K = \mathrm{Aut}(V)$ the automorphism group of $V$. Then $\tau^\sigma(z) = Q(z)\tau(-z^{-1})$. Observe that in the case $r = 1$, $\tau$ equals $-\tau^\sigma$.

Denote by

$$E = \tau \text{ and } F = \tau^\sigma$$

and let $X_E \in \mathfrak{k}_1$ and $X_F \in \mathfrak{k}_{-1}$ such that

$$E = d\kappa(X_E)1 \text{ and } F = d\kappa(X_F)Q.$$

Then, there is $a \in V$ such that for every $t \in \mathbb{C}$, $\exp(tX_F)$ is the translation $\tau_{ta} : V \to V, v \mapsto v+ta$ or a preimage $\tilde{\tau_{ta}}$ of such translation by the covering map. Furthermore, since $E = \kappa(\sigma)F$, then $\exp(tX_E)$ equals $j \circ \tau_{ta} \circ j$ or $\sigma\tilde{\tau_{ta}}\sigma$. Since for $z \in V$,

$$F(z) = \frac{d}{\mathrm{dt}}_{|t=0} Q(z+ta) = DQ(z)(a) = 2Q(z)tr(z^{-1}a)$$

then $a = -\frac{1}{2}e$, where $e$ is the unit element in $V$, and, conformally to the notations in [FK94] page 209 for the elements of the Kantor-Koecher-Tits algebra $\mathfrak{k}$,

$$X_F = (-\frac{1}{2}e, 0, 0) := (u_2, T_2, v_2)$$

and

$$X_{\tilde{E}} = (0, 0, -\frac{1}{2}e) := (u_1, T_1, v_1)$$

and it follows that

$$[X_{\tilde{E}}, X_{\tilde{F}}] = (0, T, 0)$$

with $T \in \mathfrak{k}_0 = \mathfrak{l}$ given by

$$T = -2(u_2 \diamond v_1) = -\frac{1}{2}(e \diamond e) = -\frac{1}{2}id_V = \frac{1}{2}\tilde{H}.$$

Now, since

$$[\tilde{H}, E] = (1-r)E \quad , \quad [\tilde{H}, F] = (r-1)F \quad , \quad [E, F] = \frac{1}{2}\tilde{H}$$

then, for $H := \frac{2}{1-r}\tilde{H}$, we have

$$[H, E] = 2E \quad , \quad [H, F] = -2F \quad , \quad [E, F] = \frac{1-r}{4}H.$$

In the special case $r = 1$, we denote by $\mathcal{V} := \mathfrak{p}_1$ and $\mathcal{V}^\sigma := \mathfrak{p}_{-1}$. and by

$$F(z) = Q(z) = z^2 \text{ and } E(z) = 1.$$

Then, for $H := -2\tilde{H}$, we have

$$[H, E] = 2E \text{ and } [H, F] = -2F.$$

we consider the Lie algebra structure on $\mathfrak{g} := \mathfrak{l} \oplus \mathcal{V} \oplus \mathcal{V}^\sigma$ such that $[E, F] = H$. The Lie algebra $\mathfrak{g}$ is then isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. This special case will be be the subject of the last section.

Recall now the real form $\mathfrak{g}_\mathbb{R}$ of $\mathfrak{g}$ which will be considered in the sequel. It has been introduced in [A11] . We fix a Euclidean real form $V_\mathbb{R}$ of the complex Jordan algebra $V$, denote by $z \mapsto \bar{z}$ the conjugation of $V$ with respect to $V_\mathbb{R}$, and then consider the involution $g \mapsto \bar{g}$ of $\mathrm{Conf}(V, Q)$ given by: $\bar{g} \cdot z = \overline{g \cdot \bar{z}}$. For $(g, \mu) \in \tilde{K}$ define

$$\overline{(g, \mu)} = (\bar{g}, \bar{\mu}), \text{ where } \bar{\mu}(z) = \overline{\mu(\bar{z})}.$$

The involution $\alpha$ defined by $\alpha(g) = \sigma\bar{g}\sigma^{-1}$ is a Cartan involution of $\tilde{K}$ and $\tilde{K}_\mathbb{R} := \{g \in \tilde{K} \mid \alpha(g) = g\}$ is a compact real form of $\tilde{K}$ and it follows that $L_\mathbb{R} := L \cap \tilde{K}_\mathbb{R}$ is a compact real form of $L$. Observe that, since for

$g \in \mathrm{Str}(V)$, $j \circ g \circ j = g'$, the adjoint of $g$ with respect to the symmetric form $(w \mid w') = \tau(w\bar{w}')$, then

$$L_{\mathbb{R}} = \{l \in L \mid s(l)s(l)' = id_V\}.$$

Let $\mathfrak{u}$ be the compact real form of $\mathfrak{g}$ such that $\mathfrak{l} \cap \mathfrak{u} = \mathfrak{l}_{\mathbb{R}}$, the Lie algebra of $L_{\mathbb{R}}$. Denote by $\mathcal{W}_{\mathbb{R}} = \mathfrak{u} \cap (i\mathfrak{u})$. Then, the real Lie algebra defined by

$$\mathfrak{g}_{\mathbb{R}} = \mathfrak{l}_{\mathbb{R}} + \mathcal{W}_{\mathbb{R}} \qquad (*)$$

is a real form of $\mathfrak{g}$ and the decomposition (*) is its Cartan decomposition.

Since the complexification of the Cartan decomposition of $\mathfrak{g}_{\mathbb{R}}$ is $\mathfrak{g} = \mathfrak{l} + \mathcal{W}$ and since $\mathcal{W}$ is a sum of two simple $\mathfrak{l}$-modules, it follows that the simple real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ is of Hermitian type. One can show that

$$\mathcal{W}_{\mathbb{R}} = \{p \in \mathcal{W} \mid \beta(p) = p\}.$$

where we defined for a polynomial $p \in \mathcal{W}$, $\bar{p} = \overline{p(\bar{z})}$, and considered the antilinear involution $\beta$ of $\mathcal{W}$ given by $\beta(p) = \kappa(\sigma)\bar{p}$. In particular, $E + F$ belongs to $\mathcal{W}_{\mathbb{R}}$.

The next table gives the classification of the simple real Lie algebras $\mathfrak{g}_{\mathbb{R}}$ obtained in this way. They are exactly the simple real Lie algebras of non Hermitian type which satisfy the condition $\mathcal{O}_{\min} \cap \mathfrak{g} \neq \emptyset$.

In case of an exceptional Lie algebra $\mathfrak{g}$, the real form $\mathfrak{g}_{\mathbb{R}}$ has been identified by computing the Cartan signature. The integer $n$ is $\geq 3$.

| $V$ | $Q$ | $\mathfrak{l}$ | $\mathfrak{g}$ | $\mathfrak{l}_{\mathbb{R}}$ | $\mathfrak{g}_{\mathbb{R}}$ |
|---|---|---|---|---|---|
| $\mathbb{C}$ | $z^2$ | $\mathbb{C}$ | $\mathfrak{sl}(2,\mathbb{C})$ | $i\mathbb{R}$ | $\mathfrak{su}(1,1)$ |
| $\mathbb{C}^n$ | $\Delta(z)^2$ | $\mathfrak{so}(n,\mathbb{C})\oplus\mathbb{C}$ | $\mathfrak{sl}(n+2,\mathbb{C})$ | $\mathfrak{so}(n)\oplus i\mathbb{R}$ | $\mathfrak{so}(n,2)$ |
| $\mathrm{Sym}(r,\mathbb{C})$ | $\det(z)^2$ | $\mathfrak{sl}(r,\mathbb{C}) \oplus \mathbb{C}$ | $\mathfrak{sp}(r,\mathbb{C})$ | $\mathfrak{su}(r) \oplus i\mathbb{R}$ | $\mathfrak{sp}(r,\mathbb{R})$ |
| $M(r,\mathbb{C})$ | $\det(z)^2$ | $\mathfrak{sl}(r,\mathbb{C})^{\oplus 2}\oplus\mathbb{C}$ | $\mathfrak{sl}(2r,\mathbb{C})$ | $\mathfrak{su}(r)^{\oplus 2}\oplus i\mathbb{R}$ | $\mathfrak{su}(r,r)$ |
| $\mathrm{Skew}(2r,\mathbb{C})$ | $\det(z)$ | $\mathfrak{sl}(2r,\mathbb{C})\oplus\mathbb{C}$ | $\mathfrak{so}(4r,\mathbb{C})$ | $\mathfrak{so}(2r)$ | $\mathfrak{so}^*(4r)$ |
| $\mathrm{Herm}(3,\mathbb{O})$ | $\det(z)^2$ | $\mathfrak{e}_6(\mathbb{C}) \oplus \mathbb{C}$ | $\mathfrak{e}_7(\mathbb{C})$ | $\mathfrak{e}_6(\mathbb{R}) \oplus i\mathbb{R}$ | $\mathfrak{e}_7(-25)$ |

From now on assume $r \neq 1$. Let $\Xi$ and $\Xi^\sigma$ be the $L$-orbits of $\tau$ and $\tau^\sigma$:

$$\Xi = \{\kappa(l)\tau \mid l \in L\} \subset \mathcal{V} \quad , \quad \Xi^\sigma = \{\kappa(l)\tau^\sigma \mid l \in L\} \subset \mathcal{V}^\sigma .$$

They are conical varieties and related by $\Xi^\sigma = \kappa(\sigma)\Xi$.

Recall that the structure group $\mathrm{Str}(V_{\mathbb{R}})$ of the Euclidean real form $V_{\mathbb{R}}$ of $V$ can be written

$$\mathrm{Str}(V_{\mathbb{R}}) = K_{\mathbb{R}} A_{\mathbb{R}} K_{\mathbb{R}}$$

where $K_{\mathbb{R}}$ =Aut($V_{\mathbb{R}}$) is the automorphism group of $V_{\mathbb{R}}$ and

$A_{\mathbb{R}} = \{P(a) \mid a \in \mathcal{R}_+\}$ with $\mathcal{R}_+ = \{a = \sum\limits_{i=1}^{r} a_i c_i \mid a_1 \geq \ldots \geq a_r > 0\}$

and where $P$ is the quadratic representation and $\{c_1, \ldots, c_r\}$ is a complete system of orthogonal primitive idempotents in $V_{\mathbb{R}}$ (see [FK94], p.112). It follows by complexification that the structure group of $V$ can be written

$$\mathrm{Str}(V) = KAK$$

where $K = \mathrm{Aut}(V)$ is the automorphism group of $V$ and

$$A = \{P(a) \mid a \in \tilde{\mathcal{R}}\}$$

with

$$\tilde{\mathcal{R}} = \{a = \sum_{i=1}^{r} a_i c_i \mid a_i \in \mathbb{C}, \mathrm{Re}(a_1) \geq \ldots \geq \mathrm{Re}(a_r) > 0, \mathrm{Im}(a_1) \geq \ldots \geq \mathrm{Im}(a_r) > 0\}.$$

Recall that $jP(a)j^{-1} = P(a)'$ the adjoint of $P(a)$ and that $P(a)' = P(\bar{a})$ and $P(a)^{-1} = P(a^{-1})$ for $a \in \tilde{\mathcal{R}}$. Using the relation

:centerline$\Delta(g \cdot v) = Det(g)^{\frac{r}{n}} \Delta(v)$ for $g \in \mathrm{Str}(V)$,

we deduce that a polynomial $\xi \in \Xi$ and a polynomial $\xi^\sigma = \kappa(\sigma)\xi \in \Xi^\sigma$ can be written

$$\xi(z) = \kappa(l_\xi)\tau(z) = \mathrm{Det}(P(a))^{-\frac{r}{n}} \tau(P(a)z) = \Delta(a)^{-2} \tau(P(a)z)$$

and

$$\xi^\sigma(z) = \kappa(l_{\xi^\sigma})\tau^\sigma(z) = \mathrm{Det}(P(a))^{\frac{r}{n}} \tau^\sigma(P(a^{-1})z) = \Delta(a)^2 \tau^\sigma(P(a^{-1})z).$$

Hence we get for each of the two orbits $\Xi$ and $\Xi^\sigma$ a coordinate system

$$X := \{P(a) \mid a = \sum_{i=1}^{r} a_i c_i \in \tilde{\mathcal{R}}\}.$$

In this coordinate system, the cocycle action of $L$ is given by the following : for $l \in L$, let $s(l) = k_1 P(a(l)) k_2$ where $s : L \to \mathrm{Str(V, Q)}$ is the restriction of the covering map $s : \tilde{K} \to \mathrm{Conf}(V, Q)$, $a(l) \in \tilde{\mathcal{R}}, k_1, k_2 \in K$, then

$$\kappa(l) : P(a) \mapsto P(a(l))^{-1} P(a) = P(P(a(l))^{-\frac{1}{2}} \cdot a)$$

Observe that $\kappa(\sigma)$ acts on these coordinates as follows:

$$\kappa(\sigma):\Xi\to\Xi^\sigma, \xi(z)=\Delta(a)^{-2}\tau(P(a)z)\mapsto\xi^\sigma(z)=\Delta(a)^2\tau^\sigma(P(a)^{-1}z).$$

Let $\alpha$ be an arbitratry real number. The group $L$ acts on the spaces $\mathcal{O}(\Xi)$ and $\mathcal{O}(\Xi^\sigma)$ of holomorphic functions on $\Xi$ and $\Xi^\sigma$ respectively by:

$$\big(\pi_\alpha(l)f\big)(\xi)=\Delta(a(l))^{2\alpha}f\big(\kappa(l)^{-(r-1)}\xi)\big)$$

and

$$\big(\pi_\alpha^\sigma(l)f\big)(\xi^\sigma)=\Delta(a(l))^{2\alpha}f\big(\kappa(l)^{-(r-1)}\xi^\sigma)\big).$$

If $\xi(z)=\Delta(a)^{-2}\tau(P(a)z)$ and $f\in\mathcal{O}(\Xi)$, we will write $f(\xi)=\phi(P(a))$.

Similarly, if $\xi^\sigma(z)=\Delta(a)^2\tau^\sigma(P(a)^{-1}z)$, and $f^\sigma\in\mathcal{O}(\Xi^\sigma)$, we will write $f^\sigma(\xi^\sigma)=\phi^\sigma(P(a))$.

In the coordinates $P(a)$ with $a\in\tilde{\mathcal{R}}$, the representations $\pi_\alpha$ and $\pi_\alpha^\sigma$ are given by

$$\tilde{\pi}_\alpha(l):\phi(P(a))\mapsto\Delta(a(l))^{2\alpha}\phi(P(P(a(l))^{-\frac{r-1}{2}}\cdot a))$$

and

$$\tilde{\pi}^\sigma(l):\phi(P(a))\mapsto\Delta(a(l))^{2\alpha}\phi(P(P(a(l))^{\frac{r-1}{2}}\cdot a)).$$

Consider now the isomorphism

$$\pi(\sigma):\mathcal{O}(\Xi)\to\mathcal{O}(\Xi^\sigma), f\mapsto\pi(\sigma)f$$

where

$$\pi(\sigma)f(\xi^\sigma)=f(\kappa(\sigma)\xi^\sigma).$$

It intertwines the representations $\pi_\alpha$ and $\pi_\alpha^\sigma$ of $L$, i.e., for all $l\in L$

$$\pi(\sigma)\pi_\alpha(l)=\pi_\alpha^\sigma(l)\pi(\sigma).$$

In the coordinates $P(a)$, the isomorphism $\pi(\sigma)$ is given by:

$$\tilde{\pi}(\sigma):\mathcal{O}(X)\to\mathcal{O}(X), \phi(P(a))\mapsto\phi(P(a^{-1})).$$

**2. $L_{\mathbb{R}}$-invariant Hilbert subspaces of $\mathcal{O}(\Xi)$ and $\mathcal{O}(\Xi^\sigma)$.** — In this section, we determine the irreducible $L_{\mathbb{R}}$-invariant Hilbert subspaces of the spaces of holomorphic functions $\mathcal{O}(\Xi)$ and $\mathcal{O}(\Xi^\sigma)$ .

For every $m \in \mathbb{Z}$ we denote by
$\mathcal{O}_{-\frac{m}{(r-1)}}(\Xi)$, $\mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi)$ and $\mathcal{O}_{-\frac{m}{(r-1)}}(\Xi^\sigma)$, $\mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma)$
the spaces of holomorphic functions $f$ on $\Xi$ and $f^\sigma$ on $\Xi^\sigma$ respectively such that for every $\lambda \in \mathbb{C}^*$,

$$f(\lambda \cdot \xi) = \lambda^{-\frac{m}{(r-1)}} f(\xi),$$

$$f(\lambda \cdot \xi) = \lambda^{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}} f(\xi)$$

and

$$f^\sigma(\lambda \cdot \xi^\sigma) = \lambda^{-\frac{m}{(r-1)}} f^\sigma(\xi^\sigma),$$

$$f^\sigma(\lambda \cdot \xi^\sigma) = \lambda^{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}} f^\sigma(\xi^\sigma).$$

These spaces are invariant under the representations $\pi_\alpha$ and respectively $\pi_\alpha^\sigma$. Observe that for $\xi \in \Xi$ given by $\xi(z) = \Delta(a)^{-2}\tau(P(a)z)$, and $\lambda \in \mathbb{C}^*$,

$$\lambda \cdot \xi(z) = \Delta(\nu \cdot a)^{-2}\tau(P(\nu \cdot a)z) \text{ with } \lambda = \nu^{-2(r-1)}.$$

It follows that for $f$ in $\mathcal{O}_{-\frac{m}{(r-1)}}(\Xi)$ or in $\mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi)$ , and for $f^\sigma$ in $\mathcal{O}_{-\frac{m}{(r-1)}}(\Xi^\sigma)$ or in $\mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma)$, their corresponding functions $\phi$ and $\phi^\sigma$ on $X$ satisfy the homogeneity properties

$$\phi(\mu \cdot P(a)) = \mu^m \phi(P(a)) \text{ or } \phi(\mu \cdot P(a)) = \mu^{m+\frac{1}{2}} \phi(P(a))$$

and

$$\phi^\sigma(\mu P(a)) = \mu^m \phi^\sigma(P(a)) \text{ or } \phi^\sigma(\mu \cdot P(a)) = \mu^{m+\frac{1}{2}} \phi^\sigma(P(a)),$$

in such a way that the correspondances

$$f(\xi) \mapsto \phi(P(a)) \text{ and } f^\sigma(\xi^\sigma) \mapsto \phi^\sigma(P(a))$$

map the spaces $\mathcal{O}_{-\frac{m}{(r-1)}}(\Xi)$ and $\mathcal{O}_{-\frac{m}{(r-1)}}(\Xi^\sigma)$ to the space

$$\mathcal{O}_m(X) := \{\phi \in \mathcal{O}(X) \mid \phi(\mu P(a)) = \mu^m \phi(P(a))\} \subset \mathcal{O}_{2m}(\mathbb{C}^r)$$

and the spaces $\mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi)$ and $\mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma)$ to the space

$$\mathcal{O}_{m+\frac{1}{2}}(X) := \{\phi \in \mathcal{O}(X) \mid \phi(\mu P(a)) = \mu^{m+\frac{1}{2}} \phi(P(a))\} \subset \mathcal{O}_{2m+1}(\mathbb{C}^r)$$

Denote by $\pi_{\alpha,m}$, $\pi_{\alpha,m+\frac{1}{2}}$ and $\pi^\sigma_{\alpha,m}$, $\pi^\sigma_{\alpha,m+\frac{1}{2}}$ the restrictions of the representations $\pi_\alpha$ and $\pi_\alpha^\sigma$ to the spaces
$\mathcal{O}_{-\frac{m}{(r-1)}}(\Xi)$, $\mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi)$ and $\mathcal{O}_{-\frac{m}{(r-1)}}(\Xi^\sigma)$, $\mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma)$,
and denote by $\tilde{\pi}_{\alpha,m}$, $\tilde{\pi}_{\alpha,m+\frac{1}{2}}$ and $\tilde{\pi}^\sigma_{\alpha,m}$, $\tilde{\pi}^\sigma_{\alpha,m+\frac{1}{2}}$ the corresponding representations on the spaces $\mathcal{O}_m(X)$ and $\mathcal{O}_{m+\frac{1}{2}}(X)$.

Since for every function $\phi \in \mathcal{O}(X)$, the function $a \mapsto \phi(P(a))$ is holomorphic on the open set $\tilde{\mathcal{R}} \subset \mathbb{C}^r$, we denote by $\widetilde{\mathcal{O}}(X)$ the set of such functions which extend to holomorphic functions on $\mathbb{C}^r$ and let $\widetilde{\mathcal{O}}(\Xi)$ and $\widetilde{\mathcal{O}}(\Xi^\sigma)$ be the spaces of the corresponding functions on the orbits $\Xi$ and $\Xi^\sigma$.

Then denote by
$\widetilde{\mathcal{O}}_m(X) = \mathcal{O}_m(X) \cap \widetilde{\mathcal{O}}(X)$, and $\widetilde{\mathcal{O}}_{m+\frac{1}{2}}(X) = \mathcal{O}_{m+\frac{1}{2}}(X) \cap \widetilde{\mathcal{O}}(X)$,
and put

$$\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}}(\Xi) = \widetilde{\mathcal{O}}(\Xi) \cap \mathcal{O}_{-\frac{m}{(r-1)}}(\Xi),$$

$$\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi) = \widetilde{\mathcal{O}}(\Xi) \cap \mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi),$$

and

$$\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}}(\Xi^\sigma) = \widetilde{\mathcal{O}}(\Xi^\sigma) \cap \mathcal{O}_{-\frac{m}{(r-1)}}(\Xi^\sigma),$$

$$\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma) = \widetilde{\mathcal{O}}(\Xi^\sigma) \cap \mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma).$$

Observe that the spaces $\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}}(\Xi)$, and $\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi)$ are stable by the representations $\pi_{\alpha,m}$ and $\pi_{\alpha,m+\frac{1}{2}}$ and that the spaces $\widetilde{\mathcal{O}}_{\frac{m}{(r-1)}}(\Xi^\sigma)$ and $\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi)$ are stable by the representations $\pi^\sigma_{\alpha,m}$ and $\pi^\sigma_{\alpha,m+\frac{1}{2}}$.

Theorem 2.1. —
(i) *For $m < 0$,*

$$\widetilde{\mathcal{O}}-\frac{m}{(r-1)}(\Xi) = \widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi) = \{0\}$$

*and*

$$\widetilde{\mathcal{O}}_{-\frac{m}{r-1}}(\Xi^\sigma) = \widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma) = \{0\}.$$

(ii) *The functions $\phi \circ P$ for $\phi$ in $\widetilde{\mathcal{O}}_m(X) \cup \widetilde{\mathcal{O}}_{m+\frac{1}{2}}(X)$ with $m \geq 0$ are polynomials on $\mathbb{C}^r$.*

(iii) *The spaces $\widetilde{\mathcal{O}}_m(X)$ and $\widetilde{\mathcal{O}}_{m+\frac{1}{2}}(X)$ with $m \in \mathbb{N}$ are finite dimensional, and the representations $\tilde{\pi}_{\alpha,2m}$, $\tilde{\pi}_{\alpha,2m+1}$ and $\tilde{\pi}^\sigma_{\alpha,2m}$, $\tilde{\pi}^\sigma_{\alpha,2m+1}$ are irreducible.*

*Proof.* (i), (ii) In fact, let $f$ in $\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}}(\Xi)$ or in $\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi)$ and $\phi$ in $\widetilde{\mathcal{O}}_m(X)$ or in $\widetilde{\mathcal{O}}_{m+\frac{1}{2}}(X)$ the corresponding function on $X$. Since the function $z \in \mathbb{C}^r \mapsto \phi(P(z))$ is holomorphic at 0, it admits a power series expansion given by

$$\phi(P(z)) = \sum_{(k_1,\ldots,k_r)\in\mathbb{N}^r} a_{k_1\ldots k_r} z_1^{k_1} \ldots z_r^{k_r}.$$

Furthermore, using the homogeneity property, one obtains

$$\lambda^{2m} \sum_{(k_1,\ldots,k_r)\in\mathbb{N}^r} a_{k_1\ldots k_r} z_1^{k_1} \ldots z_r^{k_r}. = \sum_{(k_1,\ldots,k_r)\in\mathbb{N}^r} a_{k_1\ldots k_r} \lambda^{k_1+\ldots+k_r} z_1^{k_1} \ldots z_r^{k_r}$$

or

$$\lambda^{2m+1} \sum_{(k_1,\ldots,k_r)\in\mathbb{N}^r} a_{k_1\ldots k_r} z_1^{k_1} \ldots z_r^{k_r}. = \sum_{(k_1,\ldots,k_r)\in\mathbb{N}^r} a_{k_1\ldots k_r} \lambda^{k_1+\ldots+k_r} z_1^{k_1} \ldots z_r^{k_r}$$

which implies that for every $\lambda$,

$$\lambda^{2m} = \lambda^{k_1+\ldots+k_r}$$

or

$$\lambda^{2m+1} = \lambda^{k_1+\ldots+k_r}$$

i.e. $k_1 + \ldots + k_r$ equals $2m$ or $2m+1$, and then $m \geq 0$ and $\phi \circ P$ is a polynomial on $\mathbb{C}^r$.

(iii) Recall (cf. [FK94)) that $s(L) = KAN$, where $K = \mathrm{Aut}(V)$, $A = \{P(a) \mid a \in \tilde{\mathcal{R}}\}$, and $N$ is the subgroup generated by the elements $\tau_0(z^{(j)} := \exp(2z^{(j)} \diamond c_j)$, with $z^{(j)} = \sum_{k=j+1}^{r} z_{jk} \in \oplus_{k=j+1}^{r} V_{jk}$ and where $V = \oplus_{j,k} V_{jk}$ is the Pierce decomposition of $V$ with respect to the complete system of primitive orthogonal idempotents $\{c_1, \ldots, c_r\}$ and where $x \diamond y = L(xy) + [L(x), L(y)]$, $L(x)$ being the left multiplication operator $V \to V, v \mapsto xv$. The group $N$ is then the unipotent radical of $s(L)$.

The subspaces

$$\{\phi \in \widetilde{\mathcal{O}}_m(X) \mid \forall n \in N, \tilde{\pi}_{\alpha,m}(n)\phi = \phi\},$$

$$\{\phi \in \widetilde{\mathcal{O}}_{m+\frac{1}{2}}(X) \mid \forall n \in N, \tilde{\pi}_{\alpha,m+\frac{1}{2}}(n)\phi = \phi\},$$

and

$$\{\phi \in \widetilde{\mathcal{O}}_m(X) \mid \forall n \in N, \tilde{\pi}^{\sigma}_{\alpha,m}(n)\phi = \phi\},$$

$$\{\phi \in \widetilde{\mathcal{O}}_{m+\frac{1}{2}}(X) \mid \forall n \in N, \tilde{\pi}^{\sigma}_{\alpha,m+\frac{1}{2}}(n)\phi = \phi\},$$

reduce to the functions $C \cdot \Delta(z)^{\frac{m}{r}}$ in the even case and to the functions $C \cdot \Delta(z)^{\frac{m}{r}+\frac{1}{2r}}$ in the odd case, hence are one dimensional. By the theorem of the highest weight (see [G08]), it follows that the spaces $\widetilde{\mathcal{O}}_m(X)$ and $\widetilde{\mathcal{O}}_{m+\frac{1}{2}}(X)$ are finite dimensional and irreducible for these representations. $\square$

We now define $L_{\mathbb{R}}$-invariant inner products on the spaces $\mathcal{O}_{-\frac{m}{(r-1)}}(\Xi)$, $\mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi)$ and $\mathcal{O}_{-\frac{m}{(r-1)}}(\Xi^\sigma)$, $\mathcal{O}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma)$. Let

$$H(z,z') = \tau(\frac{1}{r}e + z\bar{z}') \quad , \quad H(z) := H(z,z) = \tau(\frac{1}{r}e + z\bar{z})$$

and

$$H_\sigma(z,z') = \tau^\sigma(\frac{1}{r}e + z\bar{z}') \quad , \quad H_\sigma(z) := H_\sigma(z,z) = \tau^\sigma(\frac{1}{r}e + z\bar{z}).$$

Proposition 2.2. — *For $l \in L_{\mathbb{R}}$*

$$|\mathrm{Det}(s(l))| = |\Delta(a(l))|^2 = 1$$

*and*

$$H(l\cdot z) = H(z) \quad , \quad H_\sigma(l\cdot z) = H_\sigma(z)$$

*where we recall $l\cdot z = s(l)\cdot z$ by definition.*

*Proof.* In fact, since $l$ belongs to $L_{\mathbb{R}}$ if and only if $ll' = \mathrm{id}_V$, it follows that for $l \in L_{\mathbb{R}}$,

$$|\mathrm{Det}(s(l))| = |\Delta(a(l))|^2 = 1,$$
$$\tau((l\cdot z)\cdot(\overline{l\cdot z})) = \tau((ll')\cdot z\bar{z}) = \tau(z\bar{z}).$$

∎

Define the two norms of a function $\phi \in \widetilde{\mathcal{O}}_m(X)$ by

$$\|\phi\|^2_m = \frac{1}{a_m}\int_{\mathbb{C}^r} |\phi(P(z))|^2 H(z)^{-(2m+r+1)}(dz)$$

and

$$\|\phi\|^2_{m,\sigma} = \frac{1}{a_{m,\sigma}}\int_{\mathbb{C}^r} |\phi^\sigma(P(z))|^2 H(z)_\sigma^{-\alpha(m)} m(dz)$$

and the two norms of a function $\phi \in \widetilde{\mathcal{O}}_{m+\frac{1}{2}}(X)$ by

$$\|\phi\|^2_{m+\frac{1}{2}} = \frac{1}{a_{m+\frac{1}{2}}}\int_{\mathbb{C}^r} |\phi(P(z))|^2 H(z)^{-(2m+2+r)}(dz)$$

and

$$\|\phi\|^2_{m+\frac{1}{2},\sigma} = \frac{1}{a_{m+\frac{1}{2},\sigma}}\int_{\mathbb{C}^r} |\phi^\sigma(P(z))|^2 H_\sigma(z)^{-\alpha(m+\frac{1}{2})} m(dz)$$

where we identify $z = (z_1, \ldots, z_r) \in \mathbb{C}^r$ with $z = \sum_{i=1}^{r} z_i c_i \in V$, $\alpha(m)$ is a suitable integer, $m(dz)$ is the Lebesgue measure and where the positive constants $a_m$, $a_{m+\frac{1}{2}}$, $a_{m,\sigma}$, $a_{m+\frac{1}{2},\sigma}$ are given by

$$a_m = \int_{\mathbb{C}^r} H(z)^{-(2m+r+1)} m(dz),$$

$$a_{m+\frac{1}{2}} = \int_{\mathbb{C}^r} H(z)^{-(2m+2+r)} m(dz),$$

and

$$a_{m,\sigma} = \int_{\mathbb{C}^r} H_\sigma(z)^{-\alpha(m)} m(dz),$$

$$a_{m+\frac{1}{2},\sigma} = \int_{\mathbb{C}^r} H_\sigma(z)^{-\alpha(m+\frac{1}{2})} m(dz).$$

Proposition 2.3. —

(i) *These norms are $L_{\mathbb{R}}$-invariant. Hence the normed spaces* $(\widetilde{\mathcal{O}}_m(X), \|.\|_{2m})$, $(\widetilde{\mathcal{O}}_{m+\frac{1}{2}}(X), \|.\|_{2m+1})$, $(\widetilde{\mathcal{O}}_m(X), \|.\|_{m,\sigma})$, $(\widetilde{\mathcal{O}}_{m+\frac{1}{2}}(X), \|.\|_{2m+1,\sigma})$ *are Hilbert subspaces of* $\widetilde{\mathcal{O}}(X)$.

(ii) *The reproducing kernels of the spaces* $\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}}(\Xi)$, $\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi)$, $\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}}(\Xi^\sigma)$, $\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma)$ *are respectively given by*

$$\tilde{\mathcal{K}}_{2m}(P(a), P(a')) = H(a, a')^{2m+r+1},$$
$$\tilde{\mathcal{K}}_{2m+1}(P(a), P(a')) = H(a, a')^{2m+2+r},$$
$$\tilde{\mathcal{K}}_{2m,\sigma}(P(a), P(a')) = H_\sigma(a, a')^{\alpha(m)},$$
$$\tilde{\mathcal{K}}_{2m+1,\sigma}(P(a), P(a')) = H_\sigma(a, a')^{\alpha(m+\frac{1}{2})}.$$

*Proof.* (i) In fact, for $l \in L_{\mathbb{R}}$,

$$\begin{aligned}
\|\pi_{\alpha,m}(l)\phi\|^2_{2m} &= \frac{1}{a_m}\int_{\mathbb{C}^r} |\pi_{\alpha,m}(l)\phi(P(z))|^2 H(z)^{-(2m+r+1)} m(dz) \\
&= \frac{1}{a_m}\int_{\mathbb{C}^r} |\Delta(a(l))^2 \phi(P(a(l))^{-\frac{r-1}{2}} \cdot z))|^2 H(z)^{-(2m+r+1)} m(dz) \\
&= \frac{1}{a_m}\int_{\mathbb{C}^r} |\phi(z')|^2 H(P(a(l))^{\frac{r-1}{2}} z')^{-(2m+r+1)} m(d(P(a(l))^{\frac{1}{2}} z')) \\
&= \frac{1}{a_m}\int_{\mathbb{C}^r} |\Delta(a(l))|^{\frac{2n}{r}} |\phi(z')|^2 H(P(a(l))^{\frac{r-1}{2}} z')^{-(2m+r+1)} m(dz') \\
&= \|\phi\|^2_m.
\end{aligned}$$

$\square$

Since the spaces $\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}}(\Xi)$ and $\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}}(\Xi^\sigma)$ are isomorphic to $\widetilde{\mathcal{O}}_m(X)$, and the spaces $\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi)$ and $\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma)$ are isomorphic to $\widetilde{\mathcal{O}}_{m+\frac{1}{2}}(X)$, the spaces

$\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}}(\Xi)$, $\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi)$ and $\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}}(\Xi^\sigma)$, $\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma)$

become invariant Hilbert subspaces of $\widetilde{\mathcal{O}}(\Xi)$ and $\widetilde{\mathcal{O}}(\Xi^\sigma)$ respectively , with reproducing kernels given by:

$$\mathcal{K}_{2m}(\xi,\xi') = \Phi(\xi,\xi')^{2m+r+1} \quad , \quad \mathcal{K}_{2m+1}(\xi,\xi') = \Phi(\xi,\xi')^{2m+r+2},$$

and

$$\mathcal{K}_{2m,\sigma}(\xi,\xi') = \Phi_\sigma(\xi,\xi')^{\alpha(m)} \quad , \quad \mathcal{K}_{2m+1,\sigma}(\xi,\xi') = \Phi(\xi,\xi')^{\alpha(m+\frac{1}{2})},$$

where

$$\Phi(\xi,\xi') = H(a,a') \quad , \quad \Phi_\sigma(\xi,\xi') = H_\sigma(a,a') \qquad (\xi = P(a), \xi' = P(a')).$$

THEOREM 2.4. — *The group $L_{\mathbb{R}}$ acts multiplicity free on the spaces $\widetilde{\mathcal{O}}(\Xi)$ and $\widetilde{\mathcal{O}}(\Xi^\sigma)$ . The irreducible $L_{\mathbb{R}}$-invariant subspaces of $\widetilde{\mathcal{O}}(\Xi)$ and of $\widetilde{\mathcal{O}}(\Xi^\sigma)$ are the spaces $\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}}(\Xi)$, $\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi)$ and respectively $\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}}(\Xi^\sigma)$, $\widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma)$, with $m \in \mathbb{N}$.*

*If $\mathcal{H} \subset \widetilde{\mathcal{O}}(\Xi)$ and $\mathcal{H}^\sigma \subset \widetilde{\mathcal{O}}(\Xi^\sigma)$ are $L_{\mathbb{R}}$-invariant Hilbert subspaces, the reproducing kernels of $\mathcal{H}$ and $\mathcal{H}^\sigma$ can be written*

$$\mathcal{K}(\xi,\xi') = \sum_{m\in\mathbb{N}} c_m \Phi(\xi,\xi')^{2m+r+1} + \sum_{m\in\mathbb{N}} c_{m+\frac{1}{2}} \Phi(\xi,\xi')^{2m+r+2}$$

*and*

$$\mathcal{K}_\sigma(\xi,\xi') = \sum_{m\in\mathbb{N}} c_{m,\sigma} \Phi_\sigma(\xi,\xi')^{\alpha(m)} + \sum_{m\in\mathbb{N}} c_{m+\frac{1}{2},\sigma} \Phi_\sigma(\xi,\xi')^{\alpha(m+\frac{1}{2})},$$

*where $(c_m)$, $(c_{m+\frac{1}{2}}$ and $(c_{m,\sigma})$,$(c_{m+\frac{1}{2},\sigma})$ are sequences of positive numbers such that the series*

$$\sum_{m\in\mathbb{N}} c_m \Phi(\xi,\xi')^{2m+r+1} + \sum_{m\in\mathbb{N}} c_{m+\frac{1}{2}} \Phi(\xi,\xi')^{2m+r+2}$$

*and*

$$\sum_{m\in\mathbb{N}} c_{m,\sigma} \Phi_\sigma(\xi,\xi')^{\alpha(m)} + \sum_{m\in\mathbb{N}} c_{m+\frac{1}{2},\sigma} \Phi_\sigma(\xi,\xi')^{\alpha(m+\frac{1}{2})}$$

*converge uniformly on compact subsets in $\Xi$ and $\Xi^\sigma$ respectively.*

In case of a weighted Bergman space there is an integral formula for the numbers $c_m$, $c_{m+\frac{1}{2}}$ and $c_{m,\sigma}$, $c_{m+\frac{1}{2},\sigma}$. For positive functions $p$ and $p_\sigma$ on $\mathbb{C}^r$, consider the subspaces $\mathcal{H} \subset \widetilde{\mathcal{O}}(\Xi)$ and $\mathcal{H}^\sigma \subset \widetilde{\mathcal{O}}(\Xi^\sigma)$ of functions $\phi$ such that

$$\|\phi\|^2 = \int_{\mathbb{C}^r} |\phi(P(a))|^2 p(a) m(da) \ < \infty,$$

and

$$\|\phi\|_\sigma^2 = \int_{\mathbb{C}^r} |\phi(P(a))|^2 p_\sigma(a) m(da) \ < \infty,$$

THEOREM 2.5. — *Let $F$ and $F_\sigma$ be positive functions on $[0, \infty[$, and define*

$$p(a) = F(H(a))H(a)$$

*and*

$$p_\sigma(a) = F_\sigma(H_\sigma(a))H_\sigma(a)$$

(i) *Then $\mathcal{H}$ and $\mathcal{H}^\sigma$ are $L_{\mathbb{R}}$-invariant.*

(ii) *For $\phi(P(a)) = \sum_{m\in\mathbb{N}} \phi_m(P(a)) + \sum_{m\in\mathbb{N}} \phi_{m+\frac{1}{2}}(P(a))$,*

$$\|\phi\|^2 = \sum_{m\in\mathbb{N}} \frac{1}{c_m} \|\phi_m\|_{2m}^2 + \sum_{m\in\mathbb{N}} \frac{1}{c_{m+\frac{1}{2}}} \|\phi_{m+\frac{1}{2}}\|_{2m+1}^2,$$

*and for $\phi^\sigma(P(a)) = \sum_{m\in\mathbb{N}} \phi_m^\sigma(P(a)) + \sum_{m\in\mathbb{N}} \phi_{m+\frac{1}{2}}^\sigma(P(a))$,*

$$\|\phi^\sigma\|_\sigma^2 = \sum_{m\in\mathbb{N}} \frac{1}{c_{m,\sigma}} \|\phi_m^\sigma\|_{2m,\sigma}^2 + \sum_{m\in\mathbb{N}} \frac{1}{c_{m+\frac{1}{2},\sigma}} \|\phi_{m+\frac{1}{2}}^\sigma\|_{2m+1,\sigma}^2,$$

*with*

$$\frac{1}{c_m} = \pi a_m \int_{[0,+\infty[} F(u) u^{2m} du, \quad \frac{1}{c_{m+\frac{1}{2}}} = \pi a_{m+\frac{1}{2}} \int_{[0,+\infty[} F(u) u^{2m+1} du,$$

$$\frac{1}{c_{m,\sigma}} = \pi a_{m,\sigma} \int_{[0,+\infty[} F_\sigma(u) u^{2m} du, \frac{1}{c_{m+\frac{1}{2},\sigma}} = \pi a_{m+\frac{1}{2},\sigma} \int_{[0,+\infty[} F_\sigma(u) u^{2m+1} du.$$

(iii) *The reproducing kernels of $\mathcal{H}$ and $\mathcal{H}^\sigma$ are given by*

$$\mathcal{K}(\xi, \xi') = \sum_{m\in\mathbb{N}} c_m \Phi(\xi, \xi')^{2m+1+r} + \sum_{m\in\mathbb{N}} c_{m+\frac{1}{2}} \Phi(\xi, \xi')^{2m+2+r}$$

*and*

$$\mathcal{K}_\sigma(\xi, \xi') = \sum_{m\in\mathbb{N}} c_{m,\sigma} \Phi_\sigma(\xi, \xi')^{\alpha(m)} + \sum_{m\in\mathbb{N}} c_{m+\frac{1}{2},\sigma} \Phi_\sigma(\xi, \xi')^{\alpha(m+\frac{1}{2})}$$

Proposition 2.6. —

$$a_m = \pi^r \frac{1}{(2m+r)\dots(2m+1)},$$

$$a_{m+\frac{1}{2}} = \pi^r \frac{1}{(2m+r+1)\dots(2m+2)}.$$

*Proof.* In fact,

$$\begin{aligned}
a_m &= \int_{\mathbb{C}^r} H(z)^{-(2m+r+1)} m(dz),\\
&= \int_{\mathbb{C}^r} (1+|z_1|^2+\dots+|z_r|^2)^{-(2m+r+1)},\\
&= \pi^r \int_{\mathbb{R}_+^r} (2\rho_1)\dots(2\rho_r)(1+\rho_1^2+\dots+\rho_r^2)^{-(2m+r+1)} d\rho_1\dots d\rho_r,\\
&= \pi^r \frac{1}{(2m+r)\dots(2m+1)}.
\end{aligned}$$

and

$$\begin{aligned}
a_{m+\frac{1}{2}} &= \int_{\mathbb{C}^r} H(z)^{-(2m+r+2)} m(dz),\\
&= \int_{\mathbb{C}^r} (1+|z_1|^2+\dots+|z_r|^2)^{-(2m+r+2)},\\
&= \pi^r \int_{\mathbb{R}_+^r} (2\rho_1)\dots(2\rho_r)(1+\rho_1^2+\dots+\rho_r^2)^{-(2m+r+2)} d\rho_1\dots d\rho_r,\\
&= \pi^r \frac{1}{(2m+r+1)\dots(2m+2)}.
\end{aligned}$$

□

**3. Representations of the Lie algebra**. —

In the sequel, we construct representations $\rho$ and $\rho^\sigma$ of the Lie algebra $\mathfrak{g}$, which will be the infinitesimal versions of the two (non unitary equivalent) minimal representations. Recall that the group $L$ acts on the spaces $\mathcal{O}(\Xi)$ and $\mathcal{O}(\Xi^\sigma)$ respectively by:

$$\big(\pi_\alpha(l)f\big)(\xi) = \Delta(a(l))^{2\alpha} f\big(\kappa(l)^{-(1-r)}\xi\big)$$

and

$$\big(\pi_\alpha^\sigma(l)f^\sigma\big)(\xi^\sigma) = \Delta(a(l))^{2\alpha} f^\sigma\big(\kappa(l)^{-(1-r)}\xi^\sigma\big).$$

It follows that $L$ acts on the space $\mathcal{O}(X)$ by:

$$\tilde{\pi}_\alpha(l) : \phi(P(a)) \mapsto \Delta(a(l))^{2\alpha}\phi(P(P(a(l))^{-\frac{1-r}{2}} \cdot a))$$

and

$$\tilde{\pi}^\sigma_\alpha(l) : \phi(P(a)) \mapsto \Delta(a(l))^{2\alpha}\phi(P(P(a(l))^{\frac{1-r}{2}} \cdot a)).$$

This leads by differentiation, to representations $d\pi_\alpha$ and $d\pi^\sigma_\alpha$ of the Lie algebra $\mathfrak{l}$ in the spaces $\mathcal{O}(\Xi)$ and $\mathcal{O}(\Xi^\sigma)$ and representations $d\tilde{\pi}_\alpha$ and $d\tilde{\pi}^\sigma_\alpha$ of the Lie algebra $\mathfrak{l}$ in the space $\mathcal{O}(X)$.

We will construct two representations $\rho$ and $\rho^\sigma$ of $\mathfrak{g} = \mathfrak{l} + \mathcal{W}$ on the spaces of finite sums

$$\mathcal{O}_{\text{fin}}(\Xi) = \sum_{m\in\mathbb{N}} \widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}}(\Xi) \oplus \sum_{m\in\mathbb{N}} \widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi),$$

and

$$\mathcal{O}_{\text{fin}}(\Xi^\sigma) = \sum_{m\in\mathbb{N}} \widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}}(\Xi^\sigma) \oplus \sum_{m\in\mathbb{N}} \widetilde{\mathcal{O}}_{-\frac{m}{(r-1)}-\frac{1}{2(r-1)}}(\Xi^\sigma)$$

respectively, such that, for all $X \in \mathfrak{l}$,

$$\rho(X) = d\tilde{\pi}_\alpha(X) \text{ and } \rho^\sigma(X) = d\tilde{\pi}^\sigma_\alpha(X).$$

We define first a representation $\rho$ of the subalgebra generated by $E, F, H$, isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.

In particular

$$\rho(H) = d\tilde{\pi}_\alpha(H) = \frac{d}{dt}\Big|_{t=0}\pi_\alpha(\exp(tH)).$$

For $\lambda \in \mathbb{C}$, denote by $l_\lambda$ the dilation $V \to V, v \mapsto \lambda v$. Since $l_\lambda = P(\sqrt{\lambda}e)$, i.e. $a(l_\lambda) = \sqrt{\lambda}e$, then , for $\lambda = \exp(-\frac{2}{1-r}t)$,

$$\begin{aligned}\pi_\alpha(\exp(tH)\phi(P(a)) &= \Delta(a(l_\lambda))^{2\alpha}\phi(P(\sqrt{\lambda}^{1-r}e)P(a)) \\ &= \exp(2\alpha\frac{r}{r-1}t)\phi(P(e^t a))\end{aligned}$$

and then

$$\rho(H)\phi(P(a)) = 2\alpha\frac{r}{r-1}(\phi\circ P)(a) + \mathcal{E}(\phi\circ P)(a)$$

where $\mathcal{E}$ is the Euler operator $(\mathcal{E}\phi)(P(a)) := \frac{d}{du}_{|u=1}\phi(P(u\cdot a))$.

We denote by $\tilde{\mathcal{O}}_{2m}(\mathbb{C}^r)$ and $\tilde{\mathcal{O}}_{2m+1}(\mathbb{C}^r)$ the subspaces of $\mathcal{O}(\mathbb{C}^r)$, images of $\widetilde{\mathcal{O}}_m(X)$ and $\widetilde{\mathcal{O}}_{m+\frac{1}{2}}(X)$ by the isomorphism $\phi \mapsto \phi\circ P$.

Let $\rho(E) : \mathcal{O}_{\text{fin}}(X) \to \mathcal{O}_{\text{fin}}(X)$ be the multiplication operator which maps $\tilde{\mathcal{O}}_m(X)$ to $\tilde{\mathcal{O}}_{m+1}(X)$ and $\tilde{\mathcal{O}}_{m+\frac{1}{2}}(X)$ to $\tilde{\mathcal{O}}_{m+1+\frac{1}{2}}(X)$ and defined by:

$$\rho(E) : \phi(P(z)) \mapsto \frac{i}{2} c \cdot \tau(z^2)\phi(P(z)),$$

and let $\rho(F) : \mathcal{O}_{\text{fin}}(X) \to \mathcal{O}_{\text{fin}}(X)$ be the differential operator which maps $\tilde{\mathcal{O}}_m(X)$ to $\tilde{\mathcal{O}}_{m-1}(X)$ and $\tilde{\mathcal{O}}_{m+\frac{1}{2}}(X$ to $\tilde{\mathcal{O}}_{m-1+\frac{1}{2}}(X)$ given by and defined by:

$$\rho(F) : \phi(P(z)) \mapsto \frac{i}{2} c \cdot \tau(\frac{\partial^2}{\partial z^2})(\phi \circ P)(z).$$

Lemma 3.1.

$$[\rho(H), \rho(E)] = 2\rho(E),$$
$$[\rho(H), \rho(F)] = -2\rho(F)$$

*and* $[\rho(E), \rho(F)] = \rho(H)$ *if and only if* $\alpha = \frac{1}{2}$ *and* $c = \frac{\sqrt{r-1}}{2}$.

*Proof.* Since

$$\rho(H)\phi(P(a)) = 2\alpha \frac{r}{r-1}(\phi \circ P)(a) + \mathcal{E}(\phi \circ P)(a)$$

then

$$\rho(H)\rho(E) : \phi(P(z)) \mapsto \frac{i}{2} c(2\alpha \frac{r}{r-1}(\tau(z^2)(\phi \circ P)(z)) + (\mathcal{E})(\tau(z^2)(\phi \circ P)(z)),$$
$$\rho(F)\rho(H) : \phi(P(z)) \mapsto \frac{i}{2} c\tau(z^2)(\mathcal{E})(2\alpha \frac{r}{r-1}(\phi \circ P)(a) + (\mathcal{E})(\phi \circ P)(a)),$$

and using the identity

$$\mathcal{E}(\tau(z^2)(\phi \circ P)(z) - \tau(z^2)\mathcal{E}(\phi \circ P)(z) = 2\tau(z^2)(\phi \circ P)(z),$$

one obtains

$$[\rho(H), \rho(E)] : \phi(P(z)) \mapsto 2\frac{i}{2}(\tau(z^2)(\phi \circ P)(z))$$

i.e.

$$[\rho(H), \rho(E)] = 2\rho(E).$$

Similarily,

$$\rho(H)\rho(F) : \phi(P(z)) \mapsto \frac{i}{2} c(2\alpha \frac{r}{r-1} + \mathcal{E})(\tau(\frac{\partial^2}{\partial z^2})(\phi \circ P)(z)),$$
$$\rho(F)\rho(H) : \phi(P(z)) \mapsto \frac{i}{2} c\tau(\frac{\partial^2}{\partial z^2})(2\alpha \frac{r}{r-1} + \mathcal{E})(\phi \circ P)(z),$$

and using the identity

$$\mathcal{E}\tau(\frac{\partial^2}{\partial z^2})(\phi\circ P)(z)-\tau(\frac{\partial^2}{\partial z^2})\mathcal{E})(\phi\circ P)(z)=-2\tau(\frac{\partial^2}{\partial z^2})(\phi\circ P)(z),$$

one obtains

$$[\rho(H),\rho(F)]:\phi(P(z))\mapsto\frac{i}{2}c(-2(\tau(\frac{\partial^2}{\partial z^2})(\phi\circ P)(z))$$

i.e.

$$[\rho(H),\rho(F)]=-2\rho(F).$$

$$\rho(E)\rho(F):\phi(P(z))\mapsto-\frac{1}{4}c^2\tau(z^2)\tau(\frac{\partial^2}{\partial z^2})(\phi\circ P)(z),$$
$$\rho(F)\rho(E):\phi(P(z))\mapsto-\frac{1}{4}c^2\tau(\frac{\partial^2}{\partial z^2})(\tau(z^2)(\phi\circ P)(z),$$

and using the identity

$$\tau(z^2)\tau(\frac{\partial^2}{\partial z^2})(\phi\circ P)(z)-\tau(\frac{\partial^2}{\partial z^2})(\tau(z^2)(\phi\circ P)(z)=(-2r-4\mathcal{E})(\phi\circ P)(z),$$

one obtains

$$[\rho(E),\rho(F)]:\phi(P(z))\mapsto c^2(\frac{1}{2}r+\mathcal{E})(\phi\circ P)(z).$$

It follows that $[\rho(E),\rho(F)]=\frac{1-r}{4}\rho(H)$ if and only if

$$\alpha=\tfrac{r-1}{4}\text{ and }c=i\tfrac{\sqrt{r-1}}{2}.$$

∎

For $p\in\mathcal{V}^\sigma$, define the multiplication operator $\rho(p):\mathcal{O}_{\text{fin}}(X)\to\mathcal{O}_{\text{fin}}(X)$ which maps $\tilde{\mathcal{O}}_m(X)$ to $\tilde{\mathcal{O}}_{m+1}(X)$ and $\tilde{\mathcal{O}}_{m+\frac{1}{2}}(X)$ to $\tilde{\mathcal{O}}_{m+1+\frac{1}{2}}(X)$ given by

$$\rho(p):\phi(P(z))\mapsto-\frac{\sqrt{r-1}}{4}p(z^2)\phi(P(z)).$$

For $p^\sigma=\kappa(\sigma)p\in\mathcal{V}$, define the differential operator $\rho(p^\sigma):\mathcal{O}_{\text{fin}}(X)\to\mathcal{O}_{\text{fin}}(X)$ which maps $\tilde{\mathcal{O}}_m(X)$ to $\tilde{\mathcal{O}}_{m-1}(X)$ and $\tilde{\mathcal{O}}_{m+\frac{1}{2}}(X$ to $\tilde{\mathcal{O}}_{m-1+\frac{1}{2}}(X)$ given by

$$\rho(p^\sigma):\phi(P(z))\mapsto-\frac{\sqrt{r-1}}{4}p(\frac{\partial^2}{\partial z^2})(\phi\circ P)(z).$$

Observe that these definitions are consistent with the definitions of $\rho(E)$ and $\rho(F)$ and that

$$\rho(\kappa(l)p) = \pi_\alpha(l)\rho(p)\pi_\alpha(l^{-1}) \text{ and } \rho(\kappa(l)p^\sigma) = \pi_\alpha(l)\rho(p^\sigma)\pi_\alpha(l^{-1}).$$

Recall that, for $X \in \mathfrak{l}$, $\rho(X) = d\pi_\alpha(X)$. Hence we get maps

$$\rho : \mathfrak{g} = \mathfrak{l} \oplus \mathcal{W} \to \mathrm{End}\big(\mathcal{O}_{\mathrm{fin}}(\Xi)\big)$$

and

$$\rho^\sigma = \tilde{\pi}(\sigma)\rho\tilde{\pi}(\sigma) : \mathfrak{g} = \mathfrak{l} \oplus \mathcal{W} \to \mathrm{End}\big(\mathcal{O}_{\mathrm{fin}}(\Xi^\sigma)\big).$$

Theorem 3.2. —

(i) *$\rho$ and $\rho^\sigma$ are representations of the Lie algebra $\mathfrak{g}$.*

(ii) *The spaces $\mathcal{O}_{\mathrm{even}}(\Xi)$ and $\mathcal{O}_{\mathrm{odd}}(\Xi)$ are invariant and irreducible under $\rho$.*

(iii) *The spaces $\mathcal{O}_{\mathrm{even}}(\Xi^\sigma)$ and $\mathcal{O}_{\mathrm{odd}}(\Xi^\sigma)$ are invariant and irreducible under $\rho^\sigma$.*

(iv) *$\rho$ and $\rho^\sigma$ are sums of two irreducible representations.*

*Proof. Proof.* (i) Since $\pi_\alpha$ is a representation of $L$, for $X, X' \in \mathfrak{l}$,

$$[\rho(X), \rho(X')] = \rho([X, X']).$$

Furthermore, one can show that, for $X \in \mathfrak{l}, p \in \mathcal{W}$,

$$[\rho(X), \rho(p)] = \rho([X, p]).$$

Since $\mathcal{V}$ and $\mathcal{V}^\sigma$ are abelian, It remains to show that, for $p \in \mathcal{V}, p' \in \mathcal{V}^\sigma$,

$$[\rho(p), \rho(p')] = \rho([p, p']).$$

Since $[\rho(E), \rho(F)] = \rho(H)$. Then, consider the map

$$\lambda : \mathcal{V} \wedge \mathcal{V}^\sigma \to \mathrm{End}\big(\mathcal{O}_{\mathrm{fin}}(\Xi),$$

defined by

$$\lambda(p \wedge p') = [\rho(p), \rho(p')] - \rho([p, p']).$$

We know that $\lambda(E \wedge F) = 0$. It follows that, for $g \in L$,

$$\lambda\big(\kappa(g)E \wedge \kappa(g)F\big) = 0.$$

Since the representations $\kappa$ of $L$ in $\mathcal{V}$ and in $\mathcal{V}^\sigma$ are irreducible, and $E$ and $F$ are highest and lowest vectors with respect to $P_{\max} := NA$ in $\mathcal{V}$ and $\mathcal{V}^\sigma$ respectively, the vector $E \wedge F$ is cyclic in $\mathcal{V} \wedge \mathcal{V}^\sigma$ for the action of $L$. Therefore $\lambda \equiv 0$.

(ii) Let $\mathcal{U} \neq \{0\}$ (resp. $\mathcal{U}' \neq \{0\}$) be a $\rho(\mathfrak{g})$-invariant subspace of $\mathcal{O}_{\text{even}}(\Xi)$ (resp.$\mathcal{O}_{\text{odd}}(\Xi)$) . Then $\mathcal{U}$ and $\mathcal{U}'$ are $\rho(\mathfrak{l})$-invariant. Since $\mathcal{O}_{\text{even}}(\Xi) = \sum_{m=0}^{\infty} \mathcal{O}_m(X)$ and as the subspaces $\mathcal{O}_m(X)$ are $\rho(\mathfrak{l})$-irreducible, then there exists $\mathcal{I} \subset \mathbb{N}$ ($\mathcal{I} \neq \emptyset$) such that $\mathcal{U} = \sum_{m\in\mathcal{I}} \mathcal{O}_m(X)$. Similarily, since $\mathcal{O}_{\text{odd}}(\Xi) = \sum_{m=0}^{\infty} \mathcal{O}_{m+\frac{1}{2}}(X)$ and as the subspaces $\mathcal{O}_{m+\frac{1}{2}}(X)$ are $\rho(\mathfrak{l})$-irreducible, then there exists $\mathcal{I}' \subset \mathbb{N}$ ($\mathcal{I}' \neq \emptyset$) such that $\mathcal{U}' = \sum_{m\in\mathcal{I}} \mathcal{O}_{m+\frac{1}{2}}(X)$. Observe that since $\rho(E)$ maps $\mathcal{O}_m(X)$ to $\mathcal{O}_{m+1}(X)$ and maps $\mathcal{O}_{m+\frac{1}{2}}(X)$ to $\mathcal{O}_{m+1+\frac{1}{2}}(X)$, it follows that if $\mathcal{U}$ (resp. $\mathcal{U}'$) contains $\mathcal{O}_m(X)$ (resp. $\mathcal{O}_{m+\frac{1}{2}}(X)$) , then it contains $\mathcal{O}_{m+1}(X)$ (resp. $\mathcal{O}_{m+1+\frac{1}{2}}(X)$) too.

Furthermore, since $\rho(F)$ maps $\mathcal{O}_m(X)$ to $\mathcal{O}_{m-1}(X)$ and maps $\mathcal{O}_{m+\frac{1}{2}}(X)$ to $\mathcal{O}_{m-1+\frac{1}{2}}(X)$, it follows that if $\mathcal{U}$ (resp. $\mathcal{U}'$) contains $\mathcal{O}_m(X)$ (resp. $\mathcal{O}_{m+\frac{1}{2}}(X)$) , then it contains $\mathcal{O}_{m-1}(X)$ (resp. $\mathcal{O}_{m-1+\frac{1}{2}}(X)$) too. Therefore $m_0 = m_0' = 0$, then $\mathcal{U} = \mathcal{O}_{\text{even}}(\Xi)$ and $\mathcal{U}' = \mathcal{O}_{\text{odd}}(\Xi)$. ▯

**4. Unitary representations of the corresponding real Lie group**. —

We consider, for two sequences $(c_m)$, $(c_{m+\frac{1}{2}})$ and similarly for two sequences $(c_{m,\sigma})$, $(c_{m+\frac{1}{2},:sigma})$ of positive numbers, an inner product on $\mathcal{O}_{\text{fin}}(\Xi)$ and an inner product on $\mathcal{O}_{\text{fin}}(\Xi^\sigma)$ such that

$$\|\phi\|^2 = \sum_{m\in\mathbb{N}} \frac{1}{c_m} \|\phi_m\|^2_{2m} + \sum_{m\in\mathbb{N}} \frac{1}{c_{m+\frac{1}{2}}} \|\phi_{m+\frac{1}{2}}\|^2_{2m+1}$$

and similarily,

$$\|\phi\|^2_\sigma = \sum_{m\in\mathbb{N}} \frac{1}{c_{m,\sigma}} \|\phi_m\|^2_{2m,\sigma} + \sum_{m\in\mathbb{N}} \frac{1}{c_{m+\frac{1}{2},\sigma}} \|\phi_{m+\frac{1}{2}}\|^2_{2m+1,\sigma},$$

for

$$\phi(P(a)) = \sum_{m\in\mathbb{N}} \phi_m(P(a)) + \sum_{m\in\mathbb{N}} \phi_{m+\frac{1}{2}}(P(a)).$$

These inner products are invariant under $L_{\mathbb{R}}$. We will determine the sequences $(c_m)$, $(c_{m+\frac{1}{2}})$ and $(c_{m,\sigma})$, $(c_{m+\frac{1}{2},:sigma})$ such that these inner products are invariant under the representations $\rho$ and $\rho^\sigma$ restricted to $\mathfrak{g}_{\mathbb{R}}$, respectively. We denote by $\mathcal{H}$ and $\mathcal{H}^\sigma$ the Hilbert space completion of $\mathcal{O}_{\text{fin}}(\Xi)$ and of $\mathcal{O}_{\text{fin}}(\Xi^\sigma)$ with respect to these inner products. We will assume $c_0 = c_{\frac{1}{2}} = c_{0,\sigma} = c_{\frac{1}{2},\sigma} = 1$.

THEOREM 4.1. —

(i) *The inner product of* $\mathcal{H}$ *is* $\mathfrak{g}_{\mathbb{R}}$*-invariant if*

$$c_m = \tfrac{1}{(2m)!} \text{ and } c_{m+\frac{1}{2}} = \tfrac{1}{(2m+1)!}.$$

(ii) *The reproducing kernel of* $\mathcal{H}$ *is given by*

$$\mathcal{K}(\xi,\xi') = H(z,z')^{r+1}\exp(H(z,z')) \qquad (\xi = P(a), \xi' = P(a')).$$

*Proof.* (i) Recall that $\mathcal{W}_{\mathbb{R}} = \{p \in \mathcal{W} \mid \beta(p) = p\}$, where $\beta$ is the conjugation of $\mathcal{W}$, introduced at the end of Section 1. Recall also that $\beta(\kappa(g)p) = \kappa\big(\alpha(g)\big)\beta(p)$.

The inner product of $\mathcal{H}$ is $\mathfrak{g}_{\mathbb{R}}$-invariant if and only if, for every $p \in \mathcal{W}$,

$$\rho(p)^* = -\rho\big(\beta(p)\big).$$

But this is equivalent to the single condition

$$\rho(F)^* = -\rho(E).$$

which is equivalent to the following two conditions:

(1) for every $\phi \in \widetilde{\mathcal{O}}_m(X), \quad \phi' \in \widetilde{\mathcal{O}}_{m+1}(X)$,

$$\frac{1}{c_{m+1}}(\rho(E)\phi \mid \phi') = -\frac{1}{c_m}(\phi \mid \rho(F)\phi'),$$

(2) for every $\phi \in \widetilde{\mathcal{O}}_{m+\frac{1}{2}}(X), \quad \phi' \in \widetilde{\mathcal{O}}_{m+1+\frac{1}{2}}(X)$,

$$\frac{1}{c_{m+1+\frac{1}{2}}}(\rho(E)\phi \mid \phi') = -\frac{1}{c_{\frac{m+1}{2}}}(\phi \mid \rho(F)\phi').$$

Recall that the norms of $\widetilde{\mathcal{O}}_m(X)$ and $\widetilde{\mathcal{O}}_{m+\frac{1}{2}}(X)$ are given by:

$$\|\phi\|^2_{2m} = \frac{1}{a_m}\int_{\mathbb{C}^r} |(\phi\circ P)(z)|^2 H(z)^{-(2m+r+1)} m(dz)$$

and

$$\|\phi\|^2_{2m+1} = \frac{1}{a_{m+\frac{1}{2}}}\int_{\mathbb{C}^r} |(\phi\circ P)(z)|^2 H(z)^{-(2m+2+r)} m(dz)$$

Then, the required conditions of invariance become

$$\frac{1}{c_{m+1}a_{m+1}}\int_{\mathbb{C}^r} \tau(z^2)(\phi\circ P)(z)\overline{(\phi'\circ P)(z)}H(z)^{-(2m+r+3)}m(dz),$$

$$= -\frac{1}{c_m a_m}\int_{\mathbb{C}^r} (\phi\circ P))(z)\overline{(\tau\Big(\frac{\partial^2}{\partial z^2}\Big)(\phi'\circ P)(z)}H(z)^{-(2m+r+1)}m(dz),$$

$$= -\frac{1}{c_m a_m}\int_{\mathbb{C}^r} (\phi\circ P))(z)(\tau\Big(\frac{\partial^2}{\partial \bar z^2}\Big)\overline{(\phi'\circ P))(z)}H(z)^{-(2m+r+1)}m(dz),$$

and

$$\frac{1}{c_{m+1+\frac{1}{2}}a_{m+1+\frac{1}{2}}}\int\limits_{\mathbb{C}^r}\tau(z^2)(\phi\circ P)(z)\overline{(\phi'\circ P)(z)}H(z)^{-(2m+4+r)}m(dz),$$

$$= -\frac{1}{c_{m+\frac{1}{2}}a_{m+\frac{1}{2}}}\int\limits_{\mathbb{C}^r}(\phi\circ P))(z)\overline{(\tau\Big(\frac{\partial^2}{\partial z^2}\Big)(\phi'\circ P)(z)}H(z)^{-(2m+2+r)}m(dz),$$

$$= -\frac{1}{c_{m+\frac{1}{2}}a_{m+\frac{1}{2}}}\int\limits_{\mathbb{C}^r}(\phi\circ P))(z)(\tau\Big(\frac{\partial^2}{\partial \bar{z}^2}\Big)\overline{(\phi'\circ P))(z)}H(z)^{-(2m+2+r)}m(dz),$$

where we used

$$\overline{(\tau\Big(\frac{\partial^2}{\partial z^2}\Big)(\phi'\circ P))(z)} = (\tau\Big(\frac{\partial^2}{\partial \bar{z}^2}\Big)\overline{(\phi'\circ P))(z)}.$$

By integrating by parts:

$$\begin{gathered}\int_{\mathbb{C}^r}(\phi\circ P))(z)\tau\Big(\frac{\partial^2}{\partial \bar{z}^2}\Big)\overline{(\phi'\circ P)(z)}H(z)^{-(2m+r+1)}m(dz)\\ = -\int_{\mathbb{C}^r}(\phi\circ P))(z)\overline{(\phi'\circ P)(z)}(\tau\Big(\frac{\partial^2}{\partial \bar{z}^2}\Big)(H(z)^{-(2m+r+1)})m(dz),\end{gathered}.$$

and

$$\begin{gathered}\int_{\mathbb{C}^r}(\phi\circ P))(z)\tau\Big(\frac{\partial^2}{\partial \bar{z}^2}\Big)\overline{(\phi'\circ P)(z)}H(z)^{-(2m+2+r)}m(dz)\\ = -\int_{\mathbb{C}^r}(\phi\circ P))(z)\overline{(\phi'\circ P)(z)}(\tau\Big(\frac{\partial^2}{\partial \bar{z}^2}\Big)(H(z)^{-(2m+2+r)})m(dz),\end{gathered}.$$

and using the relations

$$\tau\Big(\frac{\partial^2}{\partial \bar{z}^2}\Big)(H(z)^{-(2m+r+1)}) = (2m+r+1)(2m+r+2)(\tau(z^2)H(z)^{-(2m+r+3)},$$

and

$$\tau\Big(\frac{\partial^2}{\partial \bar{z}^2}\Big)(H(z)^{-(2m+2+r)}) = (2m+r+2)(2m+r+3)(\tau(z^2)H(z)^{-(2m+4+r)},$$

the invariance conditions can be written

$$\frac{1}{c_{m+1}a_{m+1}}\int\limits_{\mathbb{C}^r}\tau(z^2)(\phi\circ P)(z)\overline{(\phi'\circ P)(z)}H(z)^{-(2m+r+3)}m(dz)$$

$$= \frac{(2m+r+1)(2m+r+2)}{c_m a_m}\int\limits_{\mathbb{C}^r}\tau(z^2)(\phi\circ P))(z)\overline{(\phi'\circ P)(z)}H(z)^{-(2m+r+3)}m(dz), cr$$

and

$$\frac{1}{c_{m+1+\frac{1}{2}}a_{m+1+\frac{1}{2}}}\int\limits_{\mathbb{C}^r}\tau(z^2)(\phi\circ P)(z)\overline{(\phi'\circ P)(z)}H(z)^{-(2m+4+r)}m(dz)$$
$$=\frac{(2m+2+r)(2m+3+r)}{c_{m+\frac{1}{2}}a_{m+\frac{1}{2}}}\int\limits_{\mathbb{C}^r}\tau(z^2)(\phi\circ P))(z)\overline{(\phi'\circ P)(z)}H(z)^{-(2m+4+r)}m(dz),cr,$$

and these are equivalent to

$$\frac{1}{c_{m+1}a_{m+1}}=(2m+r+1)(2m+r+2)\frac{1}{c_m a_m}$$

and

$$\frac{1}{c_{m+1+\frac{1}{2}}a_{m+1+\frac{1}{2}}}=(2m+2+r)(2m+3+r)\frac{1}{c_{m+\frac{1}{2}}a_{m+\frac{1}{2}}}.$$

Since from Proposition 2.6,

$$a_m=\pi^r\frac{1}{(2m+r)\dots(2m+1)},$$

and

$$a_{m+\frac{1}{2}}=\pi^r\frac{1}{(2m+r+1)\dots(2m+2)},$$

it follows that

$$\frac{c_{m+1}}{(2m+r+2)\dots(2m+3)}=\frac{c_m}{(2m+r+1)(2m+r+2)(2m+r)\dots(2m+1)}$$

and

$$\frac{c_{m+1+\frac{1}{2}}}{(2m+r+3)\dots(2m+4)}=\frac{c_{m+\frac{1}{2}}}{(2m+2+r)(2m+r+3)(2m+r+1)\dots(2m+2)}$$

i.e.

$$c_{m+1}=\frac{1}{(2m+1)(2m+2)}c_m$$

and

$$c_{m+1+\frac{1}{2}}=\frac{1}{(2m+2)(2m+3)}c_{m+\frac{1}{2}}.$$

It follows that

$$c_m=\tfrac{1}{(2m)!}\text{ and }c_{m+\frac{1}{2}}=\tfrac{1}{(2m+1)!}.$$

(ii) By Theorem 2.5 the reproducing kernel of $\mathcal{H}$ is given by

$$
\begin{aligned}
\mathcal{K}(\xi,\xi') &= \sum_{m\in\mathbb{N}} c_m H(z,z')^{2m+r+1} + \sum_{m\in\mathbb{N}} c_{m+\frac{1}{2}} H(z,z')^{2m+2+r} \\
&= \sum_{m\in\mathbb{N}} \frac{1}{(2m)!} H(z,z')^{2m+r+1} + \sum_{m\in\mathbb{N}} \frac{1}{(2m+1)!} H(z,z')^{2m+2+r},. \\
&= H(z,z')^{r+1}\exp(H(z,z')), \\
&= e(1+tr(z\bar{z}'))^{r+1}\exp(tr(z\bar{z}'))
\end{aligned}
$$

∎

A similar result can be obtained for the representation $\rho^\sigma$. One needs to calculate the constants $a_{m,\sigma}$ and $a_{m+\frac{1}{2},\sigma}$ and then determine the suitable numbers $\alpha(m)$.

In the following, we will see that the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}^\sigma$ are weighted Bergman spaces. It means that the norm of $\phi\in\mathcal{H}$ and the norm of $\phi\in\mathcal{H}^\sigma$ are given by an integral of $|\phi|^2$ with respect to positive weights.

THEOREM 4.2. — *For $\phi\in\mathcal{H}$,*

$$\|\phi\|^2 = \int_{\mathbb{C}^r} |\phi(P(z)|^2 p(z) m(dz),$$

*with*

$$p(z) = F(H(z))H(z) = H(z)^{r+1}\exp(-H(z)) = \frac{1}{e}(1+tr(z\bar{z}))^{r+1}e^{-tr(z\bar{z})}.$$

*The integral is absolutely convergent.*

*Proof.*

a) In fact, from Theorem 2.5 it follows that if the norms of $\mathcal{H}$ and $\mathcal{H}_\sigma$ are respectively given by:

$$\|\phi\|^2 = \int_{\mathbb{C}^r} |\phi(P(a))|^2 p(a) m(da) \ < \infty$$

and

$$\|\phi\|_\sigma^2 = \int_{\mathbb{C}^r} |\phi(P(a))|^2 p_\sigma(a) m(da) \ < \infty,$$

with $F$ and $F_\sigma$ positive functions on $[0,\infty[$, and

$$p(a) = F(H(a))H(a) \quad , \quad p_\sigma(a) = F_\sigma(H_\sigma(a))H_\sigma(a),$$

then

$$\frac{1}{c_m a_m} = \pi \int_{[0,+\infty[} F(u) u^{2m} du,$$

$$\frac{1}{c_{m+\frac{1}{2}} a_{m+\frac{1}{2}}} = \pi \int_{[0,+\infty[} F(u) u^{2m+1} du,$$

and

$$\frac{1}{c_{m,\sigma} a_{m,\sigma}} = \pi \int_{[0,+\infty[} F_\sigma(u) u^{\alpha(m)} du,$$

$$\frac{1}{c_{m+\frac{1}{2},\sigma} a_{m+\frac{1}{2},\sigma}} = \pi \int_{[0,+\infty[} F_\sigma(u) u^{\alpha(m+\frac{1}{2})} du.$$

Since

$$\frac{1}{a_m c_m} = (2m)! \frac{1}{\pi^r} (2m+r) \dots (2m+1) = \frac{1}{\pi^r} (2m+r)!,$$

and

$$\frac{1}{a_{m+\frac{1}{2}} c_{m+\frac{1}{2}}} = (2m+1)! \frac{1}{\pi^r} (2m+r+1) \dots (2m+2) = \frac{1}{\pi^r} (2m+1+r)!,$$

it follows that

$$\begin{aligned} \frac{1}{a_m c_m} &= \frac{1}{\pi^r} (1)_{2m+r} \\ &= C\Gamma(2m+r+1) \\ &= C \int_0^\infty u^{2m+r} e^{-u} du. \end{aligned}$$

and

$$\begin{aligned} \frac{1}{a_{m+\frac{1}{2}} c_{m+\frac{1}{2}}} &= \frac{1}{\pi^r} (1)_{2m+1+r} \\ &= C\Gamma(2m+r+2) \qquad . \\ &= C \int_0^\infty u^{2m+1+r} e^{-u} du. \end{aligned}$$

By inversion of the Mellin transform, one obtains

$$F(u) = u^r e^{-u}$$

and then

$$p(z) = H(z)^{r+1} \exp(-H(z)) = \frac{1}{e} (1 + tr(z\bar{z}))^{r+1} e^{-tr(z\bar{z})}.$$

b) Let us consider the weighted Bergman space $\mathcal{H}^1$ whose norm is given by

$$\|\phi\|_1^2 = \int_{\mathbb{C}^r} |\phi(P(z))|^2 |p(z)| m(dz).$$

By Theorem 2.5,

$$\|\phi\|_1^2 = \sum_{m=0}^{\infty} \frac{1}{c_m^1} \|\phi_m\|_{2m}^2 + \sum_{m=0}^{\infty} \frac{1}{c_{m+\frac{1}{2}}^1} \|\phi_m\|_{2m+1}^2,$$

with

$$\frac{1}{a_m c_m^1} = C \int_0^{\infty} |F(u)| u^{2m} du$$

and

$$\frac{1}{a_{m+\frac{1}{2}} c_{m+\frac{1}{2}}^1} = C \int_0^{\infty} |F(u)| u^{2m+1} du$$

Obviously $c_m^1 \le c_m$, and $c_{m+\frac{1}{2}}^1 \le c_{m+\frac{1}{2}}$, therefore $\mathcal{H}^1 \subset \mathcal{H}$. We will show that $\mathcal{H} \subset \mathcal{H}^1$. For that we will prove that there is a constant $A$ such that

$$c_m \le A \cdot c_m^1 \text{ and } c_{m+\frac{1}{2}} \le A \cdot c_{m+\frac{1}{2}}^1.$$

Since $F(u) \ge 0$, then for $u_0 \ne 0$,

$$\int_0^{\infty} |F(u)| u^{2m} du \le \int_0^{\infty} F(u) u^{2m} du + 2 \int_0^{u_0} |F(u)| u^{2m} du$$

and

$$\int_0^{\infty} |F(u)| u^{2m+1} du \le \int_0^{\infty} F(u) u^{2m+} du + 2 \int_0^{u_0} |F(u)| u^{2m+1} du.$$

Hence

$$\frac{1}{c_m^1} \le \frac{1}{c_m} + 2 a_m u_0^{2m} \int_0^{u_0} |F(u)| du$$

and

$$\frac{1}{c_{m+\frac{1}{2}}^1} \le \frac{1}{c_{m+\frac{1}{2}}} + 2 a_{m+\frac{1}{2}} u_0^{2m+1} \int_0^{u_0} |F(u)| du.$$

It follows that the sequences $a_m c_m u_0^{2m}$ and $a_{m+\frac{1}{2}} c_{m+\frac{1}{2}} u_0^{2m+1}$ are bounded. Therefore there is a constant $A$ such that

$$\frac{1}{c_m^1} \le A \frac{1}{c_m}, \text{ and} \frac{1}{c_{m+\frac{1}{2}}^1} \le A \frac{1}{c_{m+\frac{1}{2}}},$$

and this implies that $\mathcal{H} \subset \mathcal{H}_1$.

A similar result can be obtained for the representation $\rho^\sigma$. One needs to calculate the constants $a_{m,\sigma}, a_{m+\frac{1}{2},\sigma}, c_{m,\sigma}, c_{m+\frac{1}{2},\sigma}$ and determine the function $F_\sigma$. ∎

Let $\tilde{G_{\mathbb{R}}}$ be the connected and simply connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{R}}$ and denote by $\tilde{L}_{\mathbb{R}}$ the subgroup of $\tilde{G}_{\mathbb{R}}$ with Lie algebra $\mathfrak{l}_{\mathbb{R}}$. It is a covering of $L_{\mathbb{R}}$. We denote by $s : \tilde{L}_{\mathbb{R}} \to L_{\mathbb{R}}, g \mapsto s(g)$ the canonical surjection.

Using Nelson's criterion, in a similar way than for Theorem 6.3 in [AF12], one obtains:

THEOREM 4.3. — *There is a unique unitary representation $\tilde{\pi}$ of $\tilde{G}_{\mathbb{R}}$ on $\mathcal{H}$ and a unique unitary representation $\tilde{\pi}^{\sigma}$ of $\tilde{G}_{\mathbb{R}}$ on $\mathcal{H}^{\sigma}$ such that $d\tilde{\pi} = \rho$ and $d\tilde{\pi}^{\sigma} = \rho^{\sigma}$.*

**5. The $\mathfrak{sl}_2(\mathbb{R})$-case.** —

In the special case $r = 1$, since $\tau^\sigma(z) = -\tau(z) = -z$, we denote by $F = Q \in \mathcal{V} := \mathfrak{p}_1$ and $E = 1 \in \mathcal{V}^\sigma := \mathfrak{p}_{-1}$. Then for $H := -2\tilde{H}$, we have

$$[H, E] = 2E \text{ and } [H, F] = -2F$$

and then consider the Lie algebra structure on $\mathfrak{g} := \mathfrak{l} \oplus \mathcal{V} \oplus \mathcal{V}^\sigma$ such that $[E, F] = H$. In this case $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ and the real form $\mathfrak{g}_\mathbb{R}$ is isomorphic to $\mathfrak{su}(1,1)$. The structure group of $V = \mathbb{C}$ is $\mathrm{Str}(V, Q) = \mathbb{C}^*$ acting by dilations $l_\lambda$, and, since for $\lambda \in \mathbb{C}^*, Q(\lambda \cdot z) = \lambda^2 Q(z)$, then $\tilde{K} = \mathrm{Conf}(V, Q)$ and $L = \mathrm{Str}(V, Q)$. The orbits $\Xi$ and $\Xi^\sigma$ are given by

$$\Xi = \{\kappa(l_z^{-1})Q \mid z \in \mathbb{C}^*\} = \{z \cdot Q \mid z \in \mathbb{C}^*\},$$

$$\Xi^\sigma = \{\kappa(l_z^{-1})1 \mid z \in \mathbb{C}^*\} = \{\frac{1}{z} \cdot 1 \mid z \in \mathbb{C}^*\}.$$

The variety $X$ is here given by $X = \{z \in \mathbb{C}^* \mid \mathrm{Re}(z) > 0\}$ and then $\widetilde{\mathcal{O}}(X) = \mathcal{O}(\mathbb{C})$, the space of holomorphic funtions on $\mathbb{C}$. Since every $\xi \in \Xi$ can be written $\xi(v) = (\sqrt{z})^{-2} Q(P(\sqrt{z})v)$ and every $\xi^\sigma \in \Xi^\sigma$ can be written $\xi^\sigma(v) = (\sqrt{z})^2 Q(P(\sqrt{z}^{-1})v)$ where $P(u) = u^2$ is the quadratic representation of $V = \mathbb{C}$, we deduce that the coordinates $P(a)$ correspond here to $P(a) = z$, i.e. $a = \sqrt{z}$ and the isomorphism $\tilde{\pi}(\sigma)$ maps $\phi(z)$ to $\phi(\frac{1}{z})$. It follows that for $m \in \mathbb{N}$

$$\widetilde{\mathcal{O}}_m(\Xi) = \mathcal{O}_{2m}(\mathbb{C}) = \{\psi \in \mathcal{O}(\mathbb{C}) \mid \phi(\mu \cdot z) = \mu^{2m}\psi(z)\} = \mathbb{C} \cdot z^{2m}$$

and

$$\widetilde{\mathcal{O}}_{m+\frac{1}{2}}(\Xi) = \mathcal{O}_{2m+1}(\mathbb{C}) = \{\psi \in \mathcal{O}(\mathbb{C}) \mid \phi(\mu \cdot z) = \mu^{2m+1}\psi(z)\} = \mathbb{C} \cdot z^{2m+1}.$$

Then $\mathcal{O}_{\mathrm{fin}}(\Xi) = \{\psi(z) = \sum_{k=0}^{n} a_k z^k \mid a_k \in \mathbb{C}\}$. The norms on the spaces $\widetilde{\mathcal{O}}_m(\Xi)$ and $\widetilde{\mathcal{O}}_{m+\frac{1}{2}}(\Xi)$ are respectively given by:

$$\|\phi\|_{2m}^2 = \frac{1}{a_m} \int_\mathbb{C} |\phi(z)|^2 H(z)^{-(2m+2)} dz,$$

$$\|\phi\|_{2m+1}^2 = \frac{1}{a_{m+\frac{1}{2}}} \int_\mathbb{C} |\phi(z)|^2 H(z)^{-(2m+3)} dz$$

where $H(z) = \tau(e + z\bar{z}) = 1 + |z|^2$ and

$$a_m = \int_\mathbb{C} (1 + |z|^2)^{-(2m+2)} dz = \pi \frac{1}{2m+1},$$

$$a_{m+\frac{1}{2}} = \int_\mathbb{C} (1 + |z|^2)^{-(2m+3)} dz = \pi \frac{1}{2m+2}.$$

The reproducing kernel of the space $(\widetilde{\mathcal{O}}_m(\Xi), \|\cdot\|_{2m})$ is given by

$$\mathcal{K}(z, z') = (1 + z\bar{z}')^{2m+2}.$$

The reproducing kernel of the space $(\widetilde{\mathcal{O}}_{m+\frac{1}{2}}(\Xi), \|\cdot\|_{2m+1})$ is given by

$$\mathcal{K}(z, z') = (1 + z\bar{z}')^{2m+3}.$$

The representation $\pi_\alpha$ of $L$ on $\widetilde{\mathcal{O}}(\Xi) = \mathcal{O}(\mathbb{C})$ is given by

$$\pi_\alpha(l_\lambda)\phi(z) = \lambda^\alpha \phi(\lambda^{-1} \cdot z).$$

It follows in particular that for $\lambda = e^{-t}$, we have $\pi_\alpha(l_\lambda)\phi(z) = e^{-t\alpha}\phi(e^t \cdot z)$, then

$$d\pi_\alpha(H)\phi(z) = -\alpha\phi(z) + \mathcal{E}\phi(z).$$

The representation $\rho$ is given by

$$\rho(E)\phi(z) = \frac{i}{2} z^2 \phi(z),$$

$$\rho(F)\phi(z) = \frac{i}{2} \frac{\partial^2}{\partial z^2} \phi(z)$$

and

$$\rho(H)\phi(z) = d\pi_\alpha(H)\phi(z).$$

Since

$$z^2 \frac{\partial^2}{\partial z^2}\phi(z) - \frac{\partial^2}{\partial z^2}(z\phi(z)) = -2\phi(z) - 4\mathcal{E}\phi(z)),$$

it follows that for $\alpha = -\frac{1}{2}$, we have

$$[\rho(E), \rho(F)] = d\pi_\alpha(H).$$

The invariance condition $\rho(F)^* = -\rho(E)$ is equivalent to the two following conditions:

$$\frac{1}{a_{m+1}c_{m+1}} \int_{\mathbb{C}} z^2 \phi(z) \overline{\phi'(z)} (1 + |z|^2)^{-(2(m+1)+2)} dz =$$

$$-\frac{1}{a_m c_m} \int_{\mathbb{C}} \phi(z) \frac{\partial^2}{\partial \bar{z}^2} \overline{\phi'(z)} (1 + |z|^2)^{-(2m+2)} dz$$

and
$$\frac{1}{a_{m+1+\frac{1}{2}}c_{m+1+\frac{1}{2}}}\int_{\mathbb{C}} z^2\phi(z)\overline{\phi'(z)}(1+|z|^2)^{-(2(m+1)+3)}dz =$$
$$-\frac{1}{a_{m+\frac{1}{2}}c_{m+\frac{1}{2}}}\int_{\mathbb{C}} \phi(z)\frac{\partial^2}{\partial\bar{z}^2}\overline{\phi'(z)}(1+|z|^2)^{-(2m+3)}dz,$$
which are equivalent to
$$\frac{1}{a_{m+1}c_{m+1}}\int_{\mathbb{C}} z^2\phi(z)\overline{\phi'(z)}(1+|z|^2)^{-(2m+4)}dz =$$
$$-\frac{1}{a_m c_m}\int_{\mathbb{C}} \phi(z)\overline{\phi'(z)}\frac{\partial^2}{\partial\bar{z}^2}(1+|z|^2)^{-(2m+2)}dz$$
and
$$\frac{1}{a_{m+1+\frac{1}{2}}c_{m+1+\frac{1}{2}}}\int_{\mathbb{C}} z^2\phi(z)\overline{\phi'(z)}(1+|z|^2)^{-(2m+5)}dz =$$
$$-\frac{1}{a_{m+\frac{1}{2}}c_{m+\frac{1}{2}}}\int_{\mathbb{C}} \phi(z)\overline{\phi'(z)}\frac{\partial^2}{\partial\bar{z}^2}(1+|z|^2)^{-(2m+3)}dz.$$
Since
$$\frac{\partial^2}{\partial\bar{z}^2}(1+|z|^2)^{-(2m+2)} = (2m+2)(2m+3)z^2(1+|z|^2)^{-(2(m+1)+2)},$$
and
$$\frac{\partial^2}{\partial\bar{z}^2}(1+|z|^2)^{-(2m+3)} = (2m+3)(2m+4)z^2(1+|z|^2)^{-(2(m+1)+3)},$$
it follows, after an integration by parts, that the invariance conditions can be written
$$\frac{1}{a_{m+1}c_{m+1}} = (2m+2)(2m+3)\frac{1}{a_m c_m}$$
and
$$\frac{1}{a_{m+1+\frac{1}{2}}c_{m+1+\frac{1}{2}}} = (2m+3)(2m+4)\frac{1}{a_{m+\frac{1}{2}}c_{m+\frac{1}{2}}}.$$
Finally, using the formulas $a_m = \pi\frac{1}{2m+1}$ and $a_{m+\frac{1}{2}} = \pi\frac{1}{2m+2}$ , we get
$$(2m+3)\frac{1}{c_{m+1}} = (2m+2)(2m+3)(2m+1)\frac{1}{c_m}$$

and

$$(2m+4)\frac{1}{c_{m+1+\frac{1}{2}}} = (2m+3)(2m+4)(2m+2)\frac{1}{c_{m+\frac{1}{2}}}$$

i.e.

$$\frac{1}{c_{m+1+\frac{1}{2}}} = (2m+3)(2m+2)\frac{1}{c_{m+\frac{1}{2}}}$$

and it follows that

$$c_{m+\frac{1}{2}} = \frac{1}{(2m+1)!}.$$

The representation $d\pi_{\frac{1}{2}} + \rho$ integrates to a unitary representation of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ in the Hilbert space representation $\mathcal{H}$, the completion of $\mathcal{O}_{\mathrm{fin}}(\Xi)$ with respect to the norm given for $\phi = \sum_{m\in\mathbb{N}} \phi_m + \sum_{m\in\mathbb{N}} \phi_{m+\frac{1}{2}}$, by

$$\|\phi\|^2 = \sum_{m\in\mathbb{N}} \frac{1}{c_m}\|\phi_m\|^2_{2m} + \sum_{m\in\mathbb{N}} \frac{1}{c_{m+\frac{1}{2}}}\|\phi_{m+\frac{1}{2}}\|^2_{2m+1},$$

and the reproducing kernel of $\mathcal{H}$ is given by:

$$\mathcal{K}(z,z') = \sum_{m\in\mathbb{N}} \frac{1}{m!}H(z,z')^{m+2} = H(z,z')^2 e^{H(z,z')}.$$

Furthermore the Hilbert space $\mathcal{H}$ is a weighted Bergman space and its norm is given by

$$\|\phi\|^2 = \frac{1}{e}\int_{\mathbb{C}} |\phi(z)|^2(1+|z|^2)^2 e^{-|z|^2} m(dz).$$

## References


[A00] D. Achab (2000),*Algèbres de Jordan de rang 4 et représentations minimales,*Advances in Mathematics, **153,** 155-183.

[A11] D. Achab (2011),*Construction process for simple Lie algebras,* Journal of Algebra, **325,** 186-204.

[AF12] D. Achab and J. Faraut (2012), *Analysis of the Brylinski-Kostant model for minimal representations, Canad. J. Math.* **64** , 721-754.

[B98] R. Brylinski (1998), *Geometric quantization of real minimal nilpotent orbits,* Symplectic geometry Differential Geom. Appl., 9—5-58.

[BK94] R. Brylinski and B. Kostant (1994), *Minimal representations, geometric quantization, and unitarity,* Proc. Nat. Acad. USA, **91,** 6026-6029.

[BK95] R. Brylinski and B. Kostant (1995), *Lagrangian models of minimal representations of $E_6$, $E_7$ and $E_8$,* in Functional Analysis on the Eve of the 21st Century. In honor of I.M. Gelfand's 80th Birthday, 13-53, Progress in Math.131, Birkhäuser.

[C79] P. Cartier (1979), *Representations of p-adic groups in* Automorphic forms, representations and $L$-functions*,* Proc. Symposia in Pure Math.—31.1, 111-155.

[FK94] J. Faraut and A. Korányi (1994), *Analysis on symmetric cones,* Oxford University Press.

[FG96] J. Faraut and S. Gindikin (1996), *Pseudo-Hermitian symmetric spaces of tube type, in* Topics in Geometry (S. Gindikin ed.). *,* Progress in non linear differential equations and their applications, **20**, 123-154, Birkhäuser.

[G08] R. Goodman (2008), *Harmonic analysis on compact symmetric spaces :* the legacy of Elie Cartan and Hermann Weyl in Groups and analysis, London Math. Soc. Lecture Note, **354,** 1-23.

[HKMO12] J. Hilgert, T. Kobayashi, J. Mollers, B. Orsted (2012), *Fock model and Segal-Bargmann transform for minimal representations of Hermitian Lie groups,*Arxiv preprint.

[M93] A.M. Mathai (1993), *A Handbook of Generalized Special Functions for Statistical and Physical Sciences,* Oxford University Press.

[P02] M. Pevzner (2002), *Analyse conforme sur les algèbres de Jordan—,* J. Austral. Math. Soc., **73,** 1-21.

[V91] D. A. Vogan (1991), *Associated varieties and unipotent representations—,* in: W. Barker and P. Sally eds., Harmonic Analysis on Reductive Groups, Birkhausser, 315-388.